Notes on Symmetric functions and the Symmetric Group

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He who can, does; he who cannot teaches.
George Bernard Shaw, *Man & Superman* (1903)
Those who can, do, those who can't, attend conferences.
Daily Telegraph 6th August, (1979)

Introduction

These are rough notes on symmetric functions and the symmetric group and are given purely as a guide. I intend to outline some of the basic properties of symmetric functions as relevant to application to problems in chemistry and physics. The partition of integers plays a key role and we shall first make remarks on partitions in order to establish notation and then go on to consider the standard symmetric functions, their definitions and their generators. This will lead to the important symmetric functions known as $S-$functions so named in honour of Schur. Important properties to be discussed will be their outer and inner multiplication and plethysm. At that stage we can start to look at specific applications.

Partitions

An *ordered* partition λ of *length* $p = \ell(\lambda)$, corresponds to an ordered set of p integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \tag{1}$$

such that

$$\lambda_1 \geq \lambda_2, \ldots \geq \lambda_p \geq 0 \tag{2}$$

Unless otherwise stated we shall mean by a partition an ordered partition. Normally we shall omit trailing zeros.

The *weight* ω_λ of a partition λ will be defined as the sum of its parts.

$$\omega_\lambda = |\, l| = \lambda_1 + \lambda_2, + \ldots + \lambda_p \tag{3}$$

If $|\, l| = n$ then λ is said to be a partition of n. We shall denot the set of partitions $\lambda \vdash n$ as $\mathcal{P}_n$ and the set of all partitions by $\mathcal{P}$. Thus

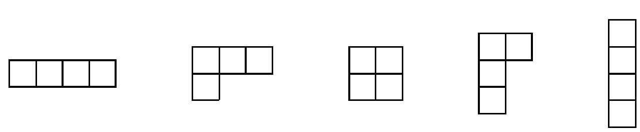
$$\mathcal{P}_4 \supseteq \{(4), (31), (2^2), (21^2), (1^4)\} \tag{4}$$

Note that the number of repetitions of a given part is often indicated by a superscript m_i where m_i is the number of parts of λ that are equal to i and will be referred to as the *multiplicity* of i in λ.

Note that in writing Eq.(4) we have given the partitions in *reverse lexicographic ordering* This ordering is such that for a pair of partitions (λ, μ) either $\lambda, \equiv \mu$ or the first non-vanishing difference $\lambda_i - \mu_i$ is *positive*.

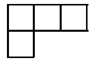
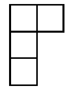
Frames of Partitions

We may associate with any partition λ a *frame* $\mathcal{F}^\lambda$ which consists of $\ell(\lambda)$, left-adjusted rows of boxes with the $i-th$ row containing λ_i boxes. Thus for $\mathcal{P}_4$ we have:-

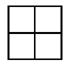


Conjugate Partitions

The *conjugate* of a partition λ is a partition λ' whose diagram is the transpose of the diagram of λ. If $\lambda' \equiv \lambda$ then the partition λ is said to be *self-conjugate*. Thus

 and 

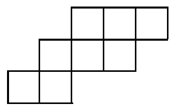
are conjugates while



is self-conjugate.

Skew Frames

Given two partitions λ and μ such that $\lambda \supset \mu$ implies that the frame F^{λ} contains the frame F^{μ}, i.e. that $\lambda_i \geq \mu_i$ for all $i \geq 1$. The difference $\rho = \lambda - \mu$ forms a *skew* frame $F^{\lambda/\mu}$. Thus, for example, the skew frame $F^{542/21}$ has the form



Note that a skew frame may consist of disconnected pieces.

Frobenius Notation for Partitions

There is an alternative notation for partitions due to Frobenius. The *diagonal* of nodes in a Ferrers-Sylvester diagram beginning at the top left-hand corner is called the *leading diagonal*. The number of nodes in the leading diagonal is called the *rank* of the partition. If r is the rank of a partition then let a_i be the number of nodes to the right of the leading diagonal in the i−th row and let b_i be the number of nodes below the leading diagonal in the i−th column. The partition is then denoted by Frobenius as

$$\begin{pmatrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_r \end{pmatrix} \tag{3.3}$$

We note that

$$a_1 > a_2 > \ldots > a_r$$
$$b_1 > b_2 > \ldots > b_r$$

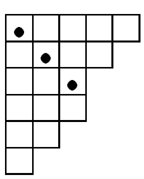
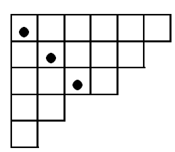
and

$$a_1 + a_2 + \ldots + a_r + b_1 + b_2 + \ldots + b_r + r = n$$

The partition conjugate to that of Eq.(3.3) is just

$$\begin{pmatrix} b_1, & b_2, & \ldots, & b_r \\ a_1, & a_2, & \ldots, & a_r \end{pmatrix} \tag{5}$$

As an example consider the partitions $(5\,4\,3^2\,2\,1)$ and $(6\,5\,4\,2\,1)$. Drawing their diagrams and marking their leading diagonal we have

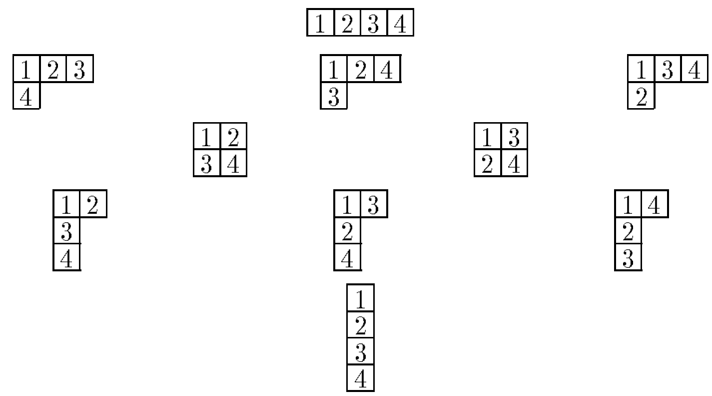
 and 

from which we deduce the respective Frobenius designations

$$\begin{pmatrix} 4 & 2 & 0 \\ 5 & 3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 3 & 1 \\ 4 & 2 & 0 \end{pmatrix}$$

Young Tableaux

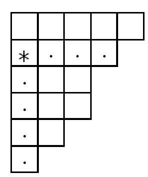
A Young tableau is an assignment of n numbers to the n cells of a frame F^λ with $\lambda \vdash n$ according to some numbering sequence. A tableau is *standard* if the assignment of the numbers $1, 2, \ldots, n$ is such that the numbers are positively increasing from left to right in rows and down columns from top to bottom. Thus for the partitions of the integer 4 we have the standard Young tableaux



We notice in the above examples that the number of standard tableaux for conjugate partitions is the same. Indeed the number of standard tableaux associated with a given frame F^λ is the *dimension* f_n^λ of an irreducible representation $\{\lambda\}$ of the symmetric group $\mathcal{S}_n$.

Hook lengths and dimensions for $\mathcal{S}_n$

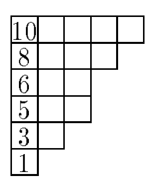
The *hook length* of a given box in a frame F^λ is the length of the right-angled path in the frame with that box as the upper left vertex. For example, the hook length of the marked box in



is 8.

Theorem 1: *To find the dimension of the representation of $\mathcal{S}_n$ corresponding to the frame F^λ, divide $n!$ by the factorial of the hook length of each box in the first column of F^λ and multiply by the difference of each pair of such hook lengths.*

Thus for the partition $(5\,4\,3^2\,2\,1)$ we have the hook lengths



and hence a dimension

$$\begin{aligned} f_{18}^{543^2 21} &= 18!\frac{2\times4\times5\times7\times9\times2\times3\times5\times7\times1\times3\times5\times2\times4\times2}{10!\times8!\times6!\times5!\times3!\times1!} \\ &= 10720710 \end{aligned}$$

It is not suggested that you check the above result by explicit enumeration!

Hook-length Product $H^{\{\lambda\}}$

The irreps $\{\lambda\}$ of S_n are indexed by the ordered partitions $\lambda \vdash N$. It is useful to define a hook-length product

$$H^{\{\lambda\}} = \prod_{(i,j)\ni\lambda} h_{ij} \tag{6}$$

where i labels rows and j columns. Note that

$$H^{\{\lambda\}} = H^{\{\lambda\prime\}} \tag{7}$$

The Frame-Robinson-Thrall Formula

The $\mathcal{S}_n$ dimensional formula may be rewritten as

$$f_n^{\lambda} = \frac{n\,!}{H^{\{\lambda\}}} \tag{8}$$

which is the celebrated result of Frame, Robinson and Thrall.

Specialisation to Two-Row Irreps of $\mathcal{C}_n$

Consider a two-part partition (p, r). It is readily seen from the definition of $H^{\{\lambda\}}$ that

$$H^{\{p,r\}} = \frac{r!\;(p+1)!}{p-r+1} \tag{9}$$

Noting that $n = p + r$ we may specialise Eq. (8) to

$$f^{\{p,r\}} = \frac{p-r+1}{p+r+1}\binom{p+r+1}{r} \tag{10}$$

In quantum chemistry the Pauli exclusion principle restricts physically realisable irreps of $\mathcal{S}_n$ to the generic type $\{\frac{N}{2}+S, \frac{N}{2}-S\}$ where N and S are the total electron number and spin respectively. In that case Eq. (10) becomes

$$f^{(N,S)} = \frac{2S+1}{N+1}\binom{N+1}{\frac{N}{2}-S} \tag{11}$$

which is sometimes called the Heisenberg formula.

Staircase Partitions

A partition of the form $(p, p-1, p-2, \ldots, 2, 1)$ is termed a *staircase partition.* Such irreps have many interesting properties.

Exercises

- Show that the $p-th$ staircase partition is of weight

$$\frac{p(p+1)}{2} \tag{12}$$

- Show that the hooklength product H^p for the $p-th$ staircase partition is

$$H^p = \prod_{i=0}^{p-1}(2i+1)^{p-i} \tag{13}$$

- Show that the $p = 18$ staircase represention is of

$$353,630,151,029,664,166,403,885,519,184,771,102,250,561,450,895,264,176,910$$
$$,003,150,360,627,549,788,542,182,043,325,740,180,684,537,821,357,203,782,730$$
$$,400,746,242,708,749,607,205,510,228,035,502,080$$

- How long would it take a supercomputer to check this result by explicit computation?

Notes on Symmetric functions and the Symmetric Group

"When a thing was new, people said, 'It is not true'. Later, when its truth became obvious, people said, 'Anyhow, it is not important' and when its importance could no longer be denied, people said, 'Anyway, it is not new'". (William James, philosopher)

Determinanntal form of the S-function

The original definition of the $S-$function was in Jacobi's determinantal form

$$s_\lambda = s_\lambda(x_1,\, x_2,\, \ldots,\, x_n) = \frac{a_{\lambda+\delta}}{a_\delta} \tag{55}$$

where λ is a partition of length $\leq n$ and $\delta = (n-1,\, n-2,\, \ldots,\, 1,\, 0)$ with

$$a_{\lambda+\delta} = det(x_i^{\lambda_j+n-j})_{1\leq i,j\leq n} \tag{56}$$

and

$$a_\delta = \prod_{1\leq i,j\leq n} (x_i - x_j) = det(x_i^{n-j}) \tag{57}$$

is the *Vandermonde determinant.*

The Vandermonde determinant is an *alternating* or *antisymmetric* function. Even powers of the Vandermonde determinant are symmetric functions. Jacobi's definition of the $S-$function is equivalent to the combinatorial definition given in Eq. (48) [cf Macdonald p23]. Both definitions have their respective merits. We shall often write in place of s_λ just $\{\lambda\}$ and assume, unless otherwise stated that the number of variables is unrestricted.

Non-standard $S-$functions

The $S-$functions are indexed by partitions. If the partitions are ordered then the $S-$function is said to be *standard.* However, from Jacobi's definition it is possible to have $S-$functions that are *non-standard* in as much as the indexing partition is not in the standard ordered form. Such non-standard $S-$functions may be transformed into a signed standard $S-$function or are null. The rules for standardising non-standard $S-$functions are often referred to as *modification rules.*

It follows from consideration of the determinant given in Eq. (56) that the relevant modification rules are:

$$\{\lambda\} = 0 \qquad \text{if } \lambda_{i+1} = \lambda_i + 1 \tag{58a}$$

$$\{\lambda_1, \lambda_2, \ldots, -\lambda_p\} = 0 \tag{58b}$$

$$\{\lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_p\} = -\{\lambda_1, \ldots, \lambda_{i+1} - 1, \lambda_i + 1, \ldots, \lambda_p\} \tag{58c}$$

Repeated application of the above three rules will reduce any non-standard $S-$function to either zero or to a signed standard $S-$function. In the process of using the above rules trailing zero parts are omitted.

Slinkies and Modification Rules

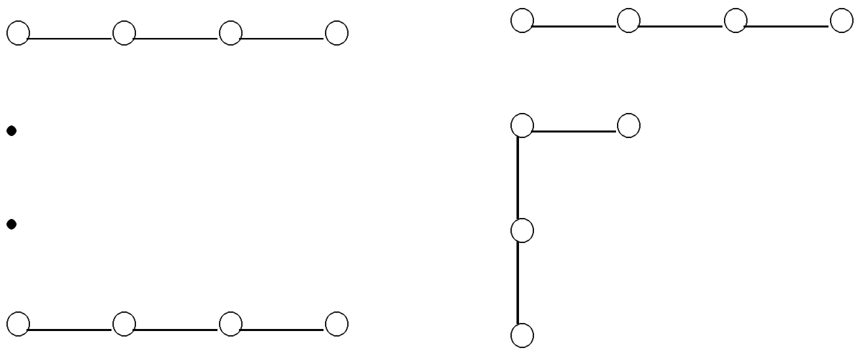
In situations involving extensive use of modification rules and in particular when one is trying to derive general formulae the use of *slinkies* can be very useful (KWY:King, Wybourne and Yang, *J. Phys. A: Math. Gen.* **22**, 4519 (1989)). (see also Chen, Garsia and Remmel, *Contemp. Math.* **34**, 109 (1984)). A slinky of length q is a diagram of q circles joined by $q-1$ links. A slinky can be folded so as to take the shape of a continuous boundary strip of a regular Young diagram, with each of the links eithehorizontal or vertical and its circles forming part of the boundary of such a diagram. The *sign* of the slinky is defined to be $(-1)^{r-1}$ where r is the number of rows occupied by the circles of the slinky, so that $r-1$ is the number of vertical links of the slinky.

The modification rules for non-standard $S-$functions can be implemented in terms of folding operations of the slinkies that make up the Young diagram as follows:

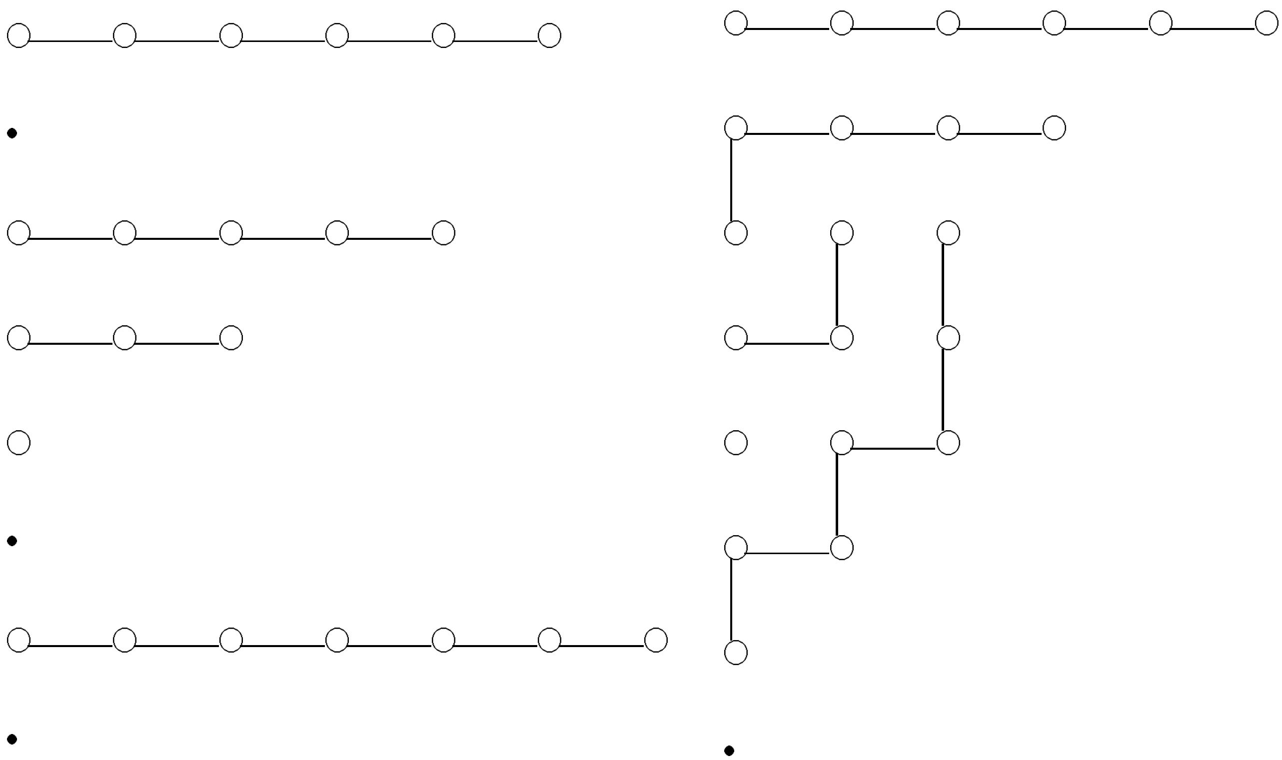
1. Draw the slinky diagram corresponding to the non-standard $S-$function $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$.
2. Successively, for $i = 1, 2, \ldots, p$, while holding the starting positions of the slinkies fixed, fold (if necessary) the $i-$th slinky of length λ_i into the shape of the unique standard continuous boundary strip such that the first i rows of the resulting diagram constitute a regular Young

diagram. If this is not possible then $\{\lambda\} = 0$. Otherwise we obtain, after folding the last slinky, the regular Young diagram corresponding to some standard $S-$function $\{\mu\}$. The final result is then $\{\lambda\} = (-1)^v\{\mu\}$ where v is the total number of vertical links in the diagram.

We illustrate the application of the method of slinkies with two examples.



$\{4004\} \quad \Rightarrow \quad \{4211\}$



$\{60531070\} \quad \Rightarrow \quad \{64333210\}$

The principal application of the slinky method is to the expansion of symmetric generating functions as a sum of $S-$functions. Thus, for example, one (KWY) can show that

$$\prod_i (1 + x_i - xi^2) = \sum_{q,r=0}^{\infty} (-1)^q f_{r+1}\{2^q 1^r\}$$

where f_{r+1} is the $(r+1)-$th Fibonacci number.

Exercises

1. Using Eqs. (58a-c) show that

$\{24\} = -\{3^2\}, \quad \{141\} = -\{321\}, \quad \{3042\} = 0, \quad \{3043\} = +\{3^2 2\}, \quad \{14-25-14\} = -\{3^3 2\}$

2. Extend the slinky algorithm to include the possibility of negative parts and then show that $\{14-25-14\} = -\{3^32\}$.
3. Use the method of slinkies to show that

$$\{60531070\} = \{643^321\} \quad \text{and} \quad \{61131090\} = 0$$

General Remarks concerning $S-$functions

The $S-$functions are symmetric functions and form an integral basis for the ring of symmetric functions and hence may be expressed in terms of the classical symmetric functions e_λ, h_λ, m_λ, f_λ. Transition matrices can be defined for taking one from members of one basis to another. The transition matrices can be expressed in terms of the Kostka matrix $K_{\lambda\mu}$ and the transposition matrix

$$J_{\lambda\mu} = \begin{cases} 1, & \text{if} \tilde{\lambda} = \mu; \\ 0. & \text{otherwise} \end{cases} \tag{59}$$

The relevant transition matrices are tabulated in Macdonald (p56). These matrices all involve integers only.

The elementary and homogeneous symmetric functions e_n and h_n are special cases of $S-$functions , namely,

$$e_n \equiv \{1^n\} \qquad h_n \equiv \{n\} \tag{60}$$

$S-$functions arise in many situations. In the next few lectures we shall explore some of their properties that are relevant to applications in physics an chemistry. To proceed to these we must first consider the Littlewood-Richardson rule and then discuss the role of $S-$functions in the character theory of the symmetric group $S(n)$ and the unitary group $U(n)$.

Skew $S-$functions

The combinatorial definition given for $S-$functions in Eq.(48) is equally valid for skew tableaux and can hence be used to define *skew* $S-$functions $s_{\lambda/\mu}(x)$ or $\{\lambda/\mu\}$. Since the $s_{\lambda/\mu}(x)$ are symmetric functions they must be expressible in terms of $S-$functions $s_\nu(x)$ such that

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu \tag{61}$$

It may be shown that the coefficients $c^\lambda_{\mu\nu}$ are necessarily non-negative integers and symmetric with respect to μ and ν. The coefficients $c^\lambda_{\mu\nu}$ are commonly referred to as *Littlewood-Richardson* coefficients.

The Littlewood-Richardson rule

The product of two $S-$functions can be written as a sum of $S-$functions, viz.

$$s_\mu . s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \tag{62}$$

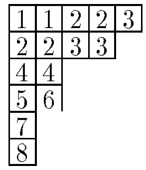
The Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ in Eqs. (61) and (62) are identical, though the summations are of course different. In both cases $|\mu| + |\nu| = |\lambda|$. A rule for evaluating the coefficients $c^\lambda_{\mu\nu}$ was given by Littlewood and Richardson in 1934 and has played a major role in all subsequent developments. The rule may be stated in various ways. We shall state it first in terms of semistandard tableaux and then also give the rule for evaluating the product given in Eq.(62) which is commonly referred to as the *outer multiplication* of $S-$functions. In each statement the concepts of a *row-word* and of a *lattice permutation* is used.

'Fred!' cried Mr Swiveller, tapping his nose, 'a word to the wise is sufficient for them - we may be good and happy without riches, Fred.'
Charles Dickens *Old Curiosity Shop* (1841).

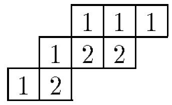
Definition 1 A word

Let T be a tableau. From T we derive a row-word or sequence $w(T)$ by reading the symbols in T from right to left (i.e. as in Arabic or Hebrew) in successive rows starting at the top row and proceeding to the bottom row

Thus for the tableau



we have the word $w(T) = 322113322446578$ and for the skew tableau



we have the word $w(T) = 11122121$.

Definition 2 A lattice permutation

A word $w = a_1a_2\ldots a_N$ in the symbols $1, 2, \ldots, n$ is said to be a lattice permutation if for $1 \le r \le N$ and $1 \le i \le n-1$, the number of occurrences of the symbol i in $a_1a_2\ldots a_r$ is not less than the number of occurrences of $i+1$.

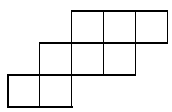
Thus the word $w(T) = 322113322446578$ is clearly not a lattice permutation whereas the word $w(T) = 11122121$ is a lattice permutation. The word $w(T) = 12122111$ is not a lattice permutation since the sub-word 12122 has more twos than ones.

Theorem 1 *The value of the coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of semistandard tableaux T of shape $F^{\lambda/\mu}$ and content ν such that $w(T)$ is a lattice permutation.*

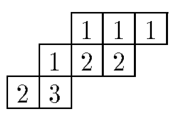
By *content* ν we mean that each tableau T contains ν_1 ones, ν_2 twos, etc.

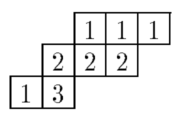
Example

Let us evaluate the coefficient $c^{\{542\}}_{\{431\}\{21\}}$. We first draw the frame $F^{\{542/21\}}$.

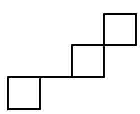


Into this frame we must inject the content of $\{431\}$ i.e. 4 ones, 3 twos and 1 three in such a way that we have a lattice permutation. We find two such numberings

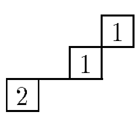


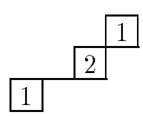


and hence $c^{\{542\}}_{\{431\}\{21\}} = 2$. Note that in the evaluation we had a choice, we could have, and indeed more simply, evaluated $c^{\{542\}}_{\{21\}\{431\}}$. In that case we would have drawn the frame $F^{\{542/431\}}$ to get



Note that in this case the three boxes are disjoint. This skew frame is to be numbered with two ones and one 2 leading to the two tableaux





verifying the previous result. Theorem 1 gives a direct method for evaluating the Littlewood-Richardson coefficients. These coefficients can be used to evaluate both skews and products. It is sometimes useful to state a procedure for directly evaluating products.

Theorem 2 *to evaluate the $S-$function product $\{\mu\}.\{\nu\}$*

1. Draw the frame F^{μ} and place ν_1 ones in the first row, ν_2 twos in the second row etc until the frame is filled with integers.

2. Draw the frame F^{ν} and inject positive integers to form a semistandard tableau such that the word formed by reading from right to left starting at the top row of the first frame and moving downwards along successive rows to the bottom row and then continuing through the second frame is a lattice permutation.

3. Repeat the above process until no further words can be constructed.

4. Each word corresponds to an $S-$function $\{\lambda\}$ where λ_1 is the number of ones, λ_2 the number of twos etc.

As an example consider the $S-$function product $\{21\} \cdot \{21\}$.

Step 1 gives the tableau

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Steps 2 and 3 lead to the eight numbered frames

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$$

Step 4 then lead to the eight words

$$\begin{array}{cccc} 112112 & 112113 & 112212 & 112213 \\ 112312 & 112314 & 112323 & 112324 \end{array}$$

from which we conclude that

$$\{21\}.\{21\} = \{42\} + \{41^2\} + \{3^2\} + 2\{321\} + \{31^3\} + \{2^3\} + \{2^21^2\}$$

I have made only one non-mathematical discovery in my life, the discovery of the exclusion principle; and that was what I was given the Nobel prize for! (Wolfgang Pauli, 1956)

Dear Professor,
I must have a serious word with you today. Are you acquainted with a certain Mr. Schrödinger, who in the year 1922 (Zeits. fur Phys.,12) described a 'bemerkenswerte Eigenschaft der Quantebahnen'? Are you acquainted with this man? What! You affirm that you know him very well, that you were even present when he did this work and that you were his accomplice in it? That is absolutely unheard of. ......
With hearty greetings, I am
Yours very faithfully

Fritz London

(Letter from F. London to E. Schrödinger 10 December 1926)

Notes on Symmetric functions and the Symmetric Group

"My association with Erwin Schrödinger was not a close one, although I spent the summer of 1927 in Zürich, with the stated purpose of working under his supervision. In fact, I spent most of my time in my room, trying to solve the Schrödinger equation for a system consisting of two helium atoms. I did not have much success, except that, as was mentioned later by John C. Slater, I formulated a determinant of the several spin-orbital functions of the individual electrons as a way of ensuring that the wave function is antisymmetric. This was a device that Slater made much use of in discussing the electronic structure of atoms and also of molecules in 1929 and 1931.". (Linus Pauling, 1956)

Relationship to the unitary group

We have explored various symmetric functions indexed by partitions and defined on sets of variables. The variables can admit many interpretations. In some instances we may choose a set of variables $1, q, q^2, \ldots, q^n$ (cf. Farmer, King and Wybourne, *J. Phys. A: Math. Gen.* **21**, 3979 (1988).) or we could even use a set of matrices. The link between $S-$functions and the character theory of groups is such that, if λ is a partition with $\ell(\lambda) \leq N$ and the eigenvalues of a group element, g, of the unitary group U_N are given by $x_j = \exp(i\phi_j)$ for $j = 1, 2, \ldots, N$ then the $S-$function

$$\begin{aligned}\{\lambda\} &= \{\lambda_1\lambda_2\ldots\lambda_N\} = s_\lambda(x) \\ &= s_\lambda(\exp(i\phi_1)\exp(i\phi_2)\ldots\exp(i\phi_N))\end{aligned} \tag{63}$$

is nothing other than the character of g in the irreducible representation of U_N conventionally designated by $\{\lambda\}$.

The Littlewood-Richardson rule gives the resolution of the Kronecker product $\{\mu\} \times \{\nu\}$ of U_N as

$$\{\mu\} \times \{\nu\} = \sum_{|\lambda|=|\mu|+|\nu|} c_{\{\mu\}.\{\nu\}}^{\{\lambda\}}\{\lambda\} \tag{64}$$

where the $c_{\{\mu\}.\{\nu\}}^{\{\lambda\}}$ are the usual Littlewood-Richardson coefficients. Equation (64) must be modified for partitions λ involving more than N parts. Here the *modification rule* is very simple. We simply discard all partitions involving more than N parts. We shall return to the unitary groups later

Reduced notation for the symmetric group

The irreps of the symmetric group $S(N)$ are uniquely labelled by the partitions $\lambda \vdash N$, there being as many irreps of $S(N)$ as there are partitions of N. Consider the following Kronecker products in $S(N)$

$$\begin{aligned}\{21\} * \{21\} &= \{3\} + \{21\} + \{1^3\} \\ \{31\} * \{31\} &= \{4\} + \{31\} + \{2^2\} + \{21^2\} \\ \{41\} * \{41\} &= \{5\} + \{41\} + \{32\} + \{31^2\}\end{aligned}$$

It is apparent that the result stabilises at $N = 4$ and in general we could write

$$\{N-1,1\} * \{N-1,1\} = \{N,0\} + \{N-1,1\} + \{N-2,2\} + \{N-2,1^2\} \tag{65}$$

The above result would hold for all N provide we apply the modification rules, Eq. (58), to any non-standard $S-$functions. Thus

$$\begin{aligned}\{21\} * \{21\} &= \{3\} + \{21\} + \{12\} + \{1^3\} \\ &= \{3\} + \{21\} + \{1^3\}\end{aligned}$$

since $\{12\} = -\{12\} = 0$.

Equation (65) could be rewritten as

$$<1> * <1> = <0> + <1> + <2> + <1^2> \tag{66}$$

The above equation is an example of the use of *reduced notation* (cf. Scharf, Thibon and Wybourne, *J. Phys. A: Math. Gen.* **26**, 7461 (1993) (STW), Butler and King, *J. Math. Phys.* **14**, 1176 (1973)(BK) and references therein.) which makes use of the fact that the symmetric group is a subgroup of the general linear group $Gl(N)$. In the reduced notation the irrep label $\{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ in $S(N)$ is replaced by $<\lambda>=<\lambda_2,\ldots,\lambda_p>$. Given any irrep $<\mu>$ in reduced notation it can be converted back into a standard irrep of $S(N)$ by prefixing it with a part $N-|\mu|$. For example, an irrep $<21>$ in reduced notation corresponds in $S(6)$ t0 $\{321\}$ or $\{921\}$ in $S(12)$. If $N-|\mu|\geq\mu_1$ then the irrep $\{N-|\mu|,\mu\}$ is assuredly a standard irrep of $S(N)$. However, if $N-|\mu|<\mu_1$ then the resulting irrep $\{N-|\mu|,\mu\}$ is non-standard and must be converted into standard form using Eq. (58).

Reduced Kronecker products for $S(N)$

BK have, following Littlewood, given the reduced Kronecker product as

$$<\lambda>*<\mu>=\sum_{\alpha,\beta,\gamma}<(\{\lambda\}/\{\alpha\}\{\beta\})\cdot(\{\mu\}/\{\alpha\}\{\gamma\})\cdot(\{\beta\}*\{\gamma\})> \qquad (67)$$

where the $\cdot$ signifies ordinary Littlewood-Richardson multiplication of the relevant $S-$function.

Exercises

1 Show that $<21>*<31>$ evaluates as

$<6>$	$+\ <52>$	$+\ <51^2>$	$+\ 4<51>$	$+\ 3<5>$	$+\ <43>$
$+\ 2<421>$	$+\ 6<42>$	$+\ <41^3>$	$+\ 6<41^2>$	$+\ 10<41>$	$+\ 5<4>$
$+\ <3^21>$	$+\ 3<3^2>$	$+\ <32^2>$	$+\ <321^2>$	$+\ 8<321>$	$+\ 11<32>$
$+\ 4<31^3>$	$+\ 12<31^2>$	$+\ 13<31>$	$+\ 5<3>$	$+\ 2<2^3>$	$+\ 3<2^21^2>$
$+\ 9<2^21>$	$+\ 8<2^2>$	$+\ <21^4>$	$+\ 6<21^3>$	$+\ 11<21^2>$	$+\ 9<21>$
$+\ 3<2>$	$+\ <1^5>$	$+\ 3<1^4>$	$+\ 4<1^3>$	$+\ 3<1^2>$	$+\ <1>$

2 Use the above result to deduce that for $S(5)$ $\{221\}*\{221\}$ evaluates as

$\{5\}$	$+\ \{41\}$	$+\ \{32\}$	$+\ \{31^2\}$	$+\ \{2^21\}$	$+\ \{21^3\}$

3 Show that in $S(8)$ $\{521\}*\{431\}$ evaluates as

$\{71\}$	$+\ 3\{62\}$	$+\ 3\{61^2\}$	$+\ 4\{53\}$	$+\ 9\{521\}$	$+\ 4\{51^3\}$
$+\ 2\{4^2\}$	$+\ 9\{431\}$	$+\ 7\{42^2\}$	$+\ 10\{421^2\}$	$+\ 3\{41^4\}$	$+\ 5\{3^22\}$
$+\ 6\{3^21^2\}$	$+\ 7\{32^21\}$	$+\ 5\{321^3\}$	$+\ \{31^5\}$	$+\ \{2^4\}$	$+\ 2\{2^31^2\}$
$+\ \{2^21^4\}$					

Kronecker products for two-row partitions

In quantum chemistry the Pauli exclusion principle restricts interest to irreps of $S(N)$ indexed by partitions into at most two parts. In terms of reduced notation two-row shapes become one-row shapes via the equivalence

$$\{N-k,k\}*\{N-\ell,\ell\}\equiv<k>*<\ell> \qquad (68)$$

From Eq. (67) we are led directly to

$$\begin{aligned}<k>*<\ell> &= \sum_{q=0}^{k}\sum_{p=0}^{\ell}<\{k-p\}\cdot\{\ell-p\}\cdot\{p-q\}> \\ &= \sum_{\lambda}c^{\nu}_{\lambda\mu}<\lambda> \qquad (69)\end{aligned}$$

The possible shapes for λ are severely constrained. The number of rows cannot exceed three. The multiplicity to be associated with a given shape λ can be readily determined by drawing the shape and then filling the cells, in accordance with the Littlewood-Richardson rule, with say $k-p$ circles $\circ$, $\ell-p$ stars $*$ and $p-q$ diamonds $\diamond$, where

$$k+\ell-p+q=\lambda_1+\lambda_2,\ldots \qquad (70)$$

Repeated cells will be marked with dots $\cdot$. Consider the shape characterised by the one-row (m), the only case relevant to quantum chemistry. A typical filling is shown below:

$\circ$	$\circ$	$\cdot$	$\cdot$	$\cdot$	$\circ$	$*$	$*$	$\cdot$	$\cdot$	$\cdot$	$*$	$\diamond$	$\diamond$	$\cdot$	$\cdot$	$\cdot$	$\diamond$

From which we can deduced that $c_{<k><\ell>}^{<m>}$ is the number of partitions of $k+\ell-m$ into two parts (p,q) with $p \geq q$ and $\ell \geq p$ leading to

$$c_{<k><\ell>}^{<m>} = \frac{1}{2}(\ell - k + m + 2) \quad \text{for} \quad k > m \tag{71a}$$

$$c_{<k><\ell>}^{<m>} = \frac{1}{2}(k + \ell - m + 2) \quad \text{for} \quad m \geq k \tag{71b}$$

and the coefficient symmetry

$$c_{<k><\ell>}^{<m>} = c_{<k><\ell>}^{<2k-m>} \tag{72}$$

Exercises

Show that

$$< 4 > * < 6 > = < 10 > + < 9 > +2 < 8 > +2 < 7 > +3 < 6 > +2 < 5 >$$
$$+ 2 < 4 > + < 3 > + < 2 >$$

and hence for $S(12)$

$$\{84\} * \{6^2\} = \{10\ 2\} + \{84\} + \{6^2\}$$

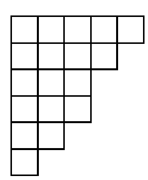
Check that the above result is dimensionally correct.

The Murnaghan-Nakayama rule for $S(N)$ characters

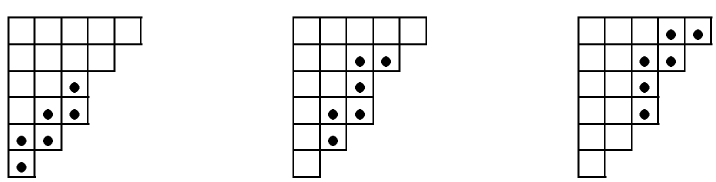
It is not my intention to give anymore than hints at methods of calculating the characters of $S(N)$ a subject well covered in the books of James and Kerber, Littlewood, Murnaghan, Macdonald, Robinson and Sagan but rather to indicate those specialisations that are of immediate application in quantum chemistry. The Murnaghan-Nakayama rule is of particular value in starting practical calculations. The key concept is that of the removal of *rim hooks* or *continuous boundary strips* from a Young frame. A rim hook is a continuous strip of cells along the boundary of the Young frame which when removed leaves a standard Young frame. The length of the strip is the total number of cells in the rim hook. We associate a *sign* with a given rim hook. If the rim hook involves v cells in the vertical direction then the sign of the rim hook is

$$sgn = (-1)^{v-1} \tag{73}$$

As an example consider the Young frame associated with the partition (543321)



Let us now mark the three permissible continuous boundary hooks of length 6 as below



In each case the 6-hook involves four rows and hence the number of vertical cells is $v = 4$ and hence the sign is $sgn = -1$.

The Murnaghan-Nakayama Algorithm The characteristic $\chi_{(\rho)}^{\{\lambda\}}$ for $S(N)$, where $\{\lambda\}$ is the irrep and (ρ) the class may be determined by

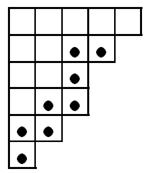
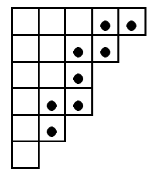
1. Draw the Young frame for the partition λ.
2. Set $i = 1$. Set $sgn = +1$.
3. While $\rho_i <> 0$ do begin

4. Remove a rim hook of length ρ_i in all possible ways that leave a standard Young frame. If this is not possible for any of the Young frames then $\chi^{\{\lambda\}}_{(\rho)} = 0$ and the algorithm is terminated.
5. A sign $sgn = sgn * newsign$ is to be associated with each new Young frame created in 3. with $newsign$ being the sign of the rim hook being removed.
6. Set $i = i + 1$
7. End
8. The characteristic $\chi^{\{\lambda\}}_{(\rho)}$ is equal to the sum of the signed units at the termination of the loop.

NB. The result is independent of the order of the removal of the rim hooks.

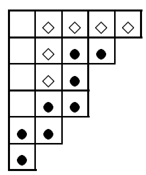
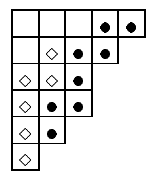
Example of $\chi^{\{543321\}}_{(864)}$

First remove a rim hook of length 8 from the Young frame as shown below

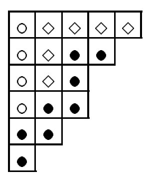
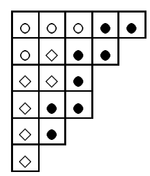



In each case the sign of the 8−hook is positive.

Now remove the 6−hook from each of the above two frames to give

Again each 6−hook has a positive sign. Now remove a 4−hook from each frame to give

The sign of each 6−hook is negative and hence each of the frames yields an overall negative sign and hence

$$\chi^{\{543321\}}_{(864)} = -2$$

The characteristics $\chi^{\{\lambda\}}_{(N)}$

The characteristics $\chi^{\{\lambda\}}_{(N)}$ constitute an important special case. By the Murnaghan-Nakayama rule there is just a single rim-hook of length N to be removed. The only possibility for a non-zero characteristic is if the frame of the partition λ is a single hook of the form $(a1^b)$ with $N = a + b$. The characteristic is thus either null or ± 1. Precisely

$$\chi^{\{\lambda\}}_{(N)} = \begin{cases} (-1)^b & \text{if } \lambda = (a+1, 1^b) \\ 0 & \text{otherwise} \end{cases}$$

The power sum symmetric functions and $S(N)$ characters

The character table of $S(N)$ is the transition matrix $M(p, s)$ that expresses power sum symmetric functions p_ρ as a linear combination of $S-$functions s_λ with $|\rho| = |\lambda| = N$. Thus

$$p_\rho = \sum_\lambda \chi^\lambda_\rho s_\lambda \tag{74}$$

We have the important special case

$$p_n = \sum_{\substack{a,b=0 \\ a+b+1=n}}^{n-1} (-1)^b s_{a+1,1^b} \tag{75}$$

Recalling that the power sum symmetric functions are multiplicative we can use Eq. (75) to compute all the characteristics associated with a given class by simple application of the Littlewood-Richardson rule. As an example consider the characteristics for the class (31) of $S(4)$. From Eq. (75) we have

$$\begin{aligned} p_3 &= \{3\} - \{21\} + \{1^3\} \\ p_1 &= \{1\} \end{aligned}$$

and hence

$$\begin{aligned} p_{31} &= (\{3\} - \{21\} + \{1^3\}) \cdot (\{1\}) \\ &= \{4\} - \{2^2\} + \{1^4\} \end{aligned}$$

showing immediately that the only non-zero characteristics associated with the class (31) are

$$\chi_{31}^{4} = +1, \qquad \chi_{31}^{2^2} = -1, \qquad \chi_{31}^{1^4} = +1$$

Exercises

1. Generalize the power sum symmetric function to

$$p_n(q;t) = \sum_{\substack{a,b=0 \\ a+b+1=n}}^{n-1} (-1)^q q^a s_{a+1,1^b}(x) \tag{76}$$

   and show that

$$p_{31}(q;x) = q^2\{4\} + (q^2-1)\{31\} - q\{2^2\} - (q-1)\{21^2\} + \{1^4\}$$

   and for $q = 1$ the $S(4)$ result is recovered. This takes one into Hecke algebras. ([KW1]King and Wybourne, *J. Phys. A: Math. Gen.* **23**, L1193(1990); [KW2]*J. Math, Phys.* **33**, 4 (1992).).

2. Construct a $q-$dependent character table for $N = 3$ and compare it with the corresponding table for $S(3)$. See [KW1].

"It did, Mr Widdershins, until quantum mechanics came along. Now everything's atoms. Reality is a fuzzy business, Mr Widdershins. I see with my eyes, which are a collection of whirling atoms, through the light, which is a collection of whirling atoms. What do I see? I see you Mr Widdershins, who are also a collection of whirling atoms. And in all this intermingling of atoms who is to know where anything starts and anything stops. It's an atomic soup we're in, Mr Widdershins. And all these quantum limbo states only collapse into one concrete reality when there is a human observer"
Pauline Melville, *The Girl with the Celestial Limb* (1991)

Notes on Symmetric functions and the Symmetric Group

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You have nothing to do but mention the quantum theory, and people will take your voice for the voice of science, and believe anything
George Bernard Shaw, *Geneva* (1938)

Murnaghan's algorithm

It is possible to produce a modification of the Murnaghan-Nakayama method to produce a systematic method for calculating characteristics χ^{λ}_{ρ} of $S(N)$. The basic idea is to choose a part ρ_i of ρ and subtract it from λ_j of λ to produce a newlist of $S-$functions of weight $N-\rho_i$

$$newlist = \sum_{j=1}^{p}\{\lambda_1,\ldots,\lambda_j-\rho_i,\ldots,\lambda_p\} \tag{77}$$

Now make newlist standard by applying the $S-$function modification rules. Next repeat the process successively until all the parts of ρ have been used. The resulting final list will involve just the $S-$function $\{0\}$ and its coefficient is the desired characteristic.

As an example consider the calculation of the characteristic χ^{311}_{221} for $S(5)$. We first subtract 2 from each part of $\{311\}$ and apply the modification rules to produce

$$\begin{aligned}\{311\} &\to \{111\}+\{3-11\}+\{31-1\}\\ &= \{111\}-\{3\}\end{aligned}$$

Now repeat the process to produce

$$\begin{aligned}\{111\}-\{3\} &\to \{-111\}+\{1-11\}+\{11-1\}-\{1\}\\ &= -2\{1\}\end{aligned}$$

Now subtract the last part 1 of $\{221\}$ to finally give

$$\chi^{311}_{221} = -2$$

This process can be readily programmed to produce a fast evaluation of arbitrary characteristics. However, it is not a suitable method for yielding formulae.

The Dimensional Formula in Reduced Notation for $S(n)$

One has the well-known Robinson-Frame-Thrall result

$$f^{\{\lambda\}} = \frac{n!}{H_\lambda} \tag{78}$$

where H_λ is the product of the hook lengths of F^λ. We can obtain a corresponding result for $\{\lambda\} = \{n-m,\mu_1,\ldots,\mu_r\}$ by considering the hook lengths for the first row of F^λ and cancelling these factors with terms in $n!$ to give in reduced notation

$$f_n^{<\mu>} = \frac{1}{H_\mu}\frac{n!}{(n-m+r)!}\prod_{i=1}^{r}(n-m-\mu_i+i) \tag{79}$$

This may be put in a form analogous to Eq. (78) by defining $\mu_i = 0$ for $i > r$ to yield

$$f_n^{<\mu>} = \frac{1}{H_\mu}\prod_{i=1}^{m}(n-m-\mu_i+i) \tag{80}$$

The advantage of Eqs (78) and (79) is that they lead directly to formulas for the dimensions as an explicit function of n.

Thus for a two-part partition $(n-m,m)$ we readily obtain the general result

$$f_n^{<m>} = \frac{n!(n-2m+1)}{m!(n-m+1)!} \tag{81}$$

and hence typically

$$f_n^{<4>} = \frac{n(n-1)(n-2)(n-7)}{24} \tag{82}$$

which is valid for all $n \geq 0$. For $n = 3,4,5,6$ we have

$$f_3^{<4>} = -1,\ f_4^{<4>} = -3,\ f_5^{<4>} = -5,\ f_6^{<4>} = -5$$

which corresponds to

$$\{-1,4\} = -\{3\},\ \{0,4\} = -\{3,1\},\ \{1,4\} = -\{3,2\},\ \{2,4\} = -\{3,3\}$$

Note that the characteristic $\chi^{\lambda}_{1^n}$ is just the dimension of the irrep $\{\lambda\}$ of $S(n)$ and hence in reduced notation

$$f_n^{<\mu>} = \chi^{\mu}_{1^n} \tag{82}$$

Raising Operators in $S(N)$

Before continuing with the characters of $S(N)$ we make an important diversion. The S-functions s_λ can be related to the homogeneous symmetric functions h_λ by writing

$$s_\lambda = |h_{\lambda_i - i + j}| \tag{83}$$

Thus, for example,

$$\begin{aligned} s_{321} &= \begin{vmatrix} h_3 & h_4 & h_5 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} \\ &= h_{321} - h_{33} - h_{411} + h_{51} \end{aligned} \tag{84}$$

where we note that $h_{-k} = 0, \quad h_0 = 1$.

Let us introduce a *raising operator* R_{ij} such that acting on a partition (λ) we have

$$R_{ij}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_p) = (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots, \lambda_p) \tag{85}$$

We can then rewrite Eq. (83) in the form

$$s_\lambda = \prod_{i<j}(1 - R_{ij})h_\lambda \tag{86}$$

Thus

$$s_{321} = (1 - R_{12})(1 - R_{13})(1 - R_{23})h_{321} \tag{87}$$

We have successively

$$\begin{aligned} &(1 - R_{23})h_{321} = h_{321} - h_{33} \\ &(1 - R_{13})[h_{321} - h_{33}] = h_{321} - h_{33} - h_{42} \\ &(1 - R_{12})[h_{321} - h_{33} - h_{42}] = h_{321} - h_{33} - h_{42} - h_{411} + h_{42} + h_{51} \\ &s_{321} = h_{321} - h_{33} - h_{411} + h_{51} \end{aligned} \tag{88}$$

in agreement with Eq. (84). Note that no modification rules are applied until the action of all the raising operators have been applied though trailing zeros may be dropped and partitions whose last part is negative.

We can define an *inverse raising operator* as

$$\prod_{i<j}(1 - R_{ij})^{-1} = \prod_{i<j}(1 + R_{ij} + R_{ij}^2 + \ldots) \tag{89}$$

We then have the inverse transformation

$$h_\lambda = \prod_{i<j}(1 + R_{ij} + R_{ij}^2 + \ldots)s_\lambda \tag{90}$$

Thus

$$h_{321} = (1 + R_{12} + R_{12}^2 + \ldots)(1 + R_{13} + R_{13}^2 + \ldots)(1 + R_{23} + R_{23}^2 + \ldots)s_{321} \tag{91}$$

We have successively

$$\begin{aligned} s_{321} &\to s_{321} + s_{33} \\ s_{321} + s_{33} &\to s_{321} + s_{33} + s_{42} \\ s_{321} + s_{33} + s_{42} &\to s_{321} + s_{33} + s_{42} + s_{411} + s_{501} + s_{6-11} + s_{42} + s_{51} + s_6 + s_{51} + s_6 \\ &\to s_{321} + s_{33} + s_{411} + 2s_{42} + 2s_{51} + s_6 \end{aligned} \tag{92}$$

where we have used $s_{501} = 0, \quad s_{6-11} = -s_6$. Recall that h_λ is a multiplicative symmetric function and hence $h_{321} = h_3h_2h_1$ and furthermore $h_n = s_n$ and hence by the Littlewood-Richardson rule

$$\begin{aligned} h_{321} &= s_3 \cdot s_2 \cdot s_1 \\ &= s_{321} + s_{33} + s_{411} + 2s_{42} + 2s_{51} + s_6 \end{aligned} \tag{93}$$

in agreement with Eq. (92).

A Reduced Notation Determinantal Form

Let us return to the determinantal expansion of s_λ in terms of h_λ as in Eq. (83). We can rewrite, anticipating reduced notation,

$$s_\lambda = s_{n-m,\mu_1,\mu_2,\ldots} \tag{94}$$

The determinantal form then becomes

$$s_{n-m,\mu_1,\mu_2,\ldots} = \begin{vmatrix} h_{n-m} & h_{n-m+1} & h_{n-m+2} & \ldots \\ h_{\mu_1-1} & h_{\mu_1} & h_{\mu_1+1} & \ldots \\ h_{\mu_2-2} & h_{\mu_2-1} & h_{\mu_2} & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix} \tag{95}$$

The above determinant can be expanded about the first row as a sum of determinants to give

$$\begin{vmatrix} h_{n-m} & h_{n-m+1} & h_{n-m+2} & \ldots \\ h_{\mu_1-1} & h_{\mu_1} & h_{\mu_1+1} & \ldots \\ h_{\mu_2-2} & h_{\mu_2-1} & h_{\mu_2} & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix} = h_{n-m} \begin{vmatrix} h_{\mu_1} & h_{\mu_1+1} & \ldots \\ \vdots & \vdots & \vdots \end{vmatrix} - h_{n-m+1} \begin{vmatrix} h_{\mu_1-1} & h_{\mu_1} & \ldots \\ \vdots & \vdots & \ldots \end{vmatrix} + \ldots \tag{96}$$

The determinant introduced in Eq. (83) readily extends to skew $S-$functions to give

$$s_{\mu/\rho} = |h_{\mu_i-\rho_i-i+j}| \tag{97}$$

Comparing Eq. (96) with Eq. (97) allows us to rewrite Eq. (95) as

$$s_{n-m,\mu_1,\mu_2,\ldots} = \sum_{r=0}^{m}(-1)^r s_{n-m+r} \cdot s_{\mu/1^r} \tag{98}$$

The dimension $f_n^{(s_\alpha \cdot s_\beta)}$ of the $S-$function product $s_\alpha \cdot s_\beta$ evaluated in $S(n)$ where $n = |\alpha| \times |\beta|$ is given by

$$f_n^{(s_\alpha \cdot s_\beta)} = \frac{(|\alpha| + |\beta|)!}{|\alpha|!|\beta|!} f^{s_\alpha} f^{s_\beta} \tag{99}$$

Putting $k = m - r$ in Eq. (98) and equating dimensions on both sides yields the identity

$$f^\lambda = f_n^{<\mu>} = \sum_{k=0}^{m}(-1)^{m-k} \binom{n}{k} f^{s_\mu/s_{1^{m-n}}} \tag{100}$$

General Results on Characters of $S(n)$

Let us write the cycle structure of a class (α) of $S(n)$ as

$$(\alpha) = (1^{\alpha_1}2^{\alpha_2}\dots k^{\alpha_k}\dots n^{\alpha_n}) \tag{101}$$

A characteristic for $S(n)$ is then written as

$$\chi^{\{\lambda\}}_{(\alpha)} = \chi^{\{\lambda_1,\lambda_2,\dots\}}_{(1^{\alpha_1}2^{\alpha_2}\dots)}$$

The Murnaghan algorithm is then essentially

$$\chi^{\{\lambda\}}_{(\alpha)} = \sum_i \chi^{\{\lambda_1,\lambda_2,\dots,\lambda_i-k,\dots\}}_{(1^{\alpha_1}2^{\alpha_2}\dots k^{\alpha_k-1}\dots)} \tag{102}$$

with the $S-$function modification rules being used as required.

We can readily arrive at a number of general results for the characters of $S(n)$. Considering the decomposition $S(m+n) \to S(m)\times S(n)$ we have

$$\chi^{\lambda}_{\sigma\tau} = \sum_{\mu\vdash n} \chi^{\lambda/\mu}_{\sigma}\chi^{\mu}_{\tau} \tag{103}$$

where $\sigma\vdash m$ and $\tau\vdash n$.

Suppose $(\tau)=(n)$ then Eq. (103) becomes

$$\chi^{\lambda}_{\sigma n} = \sum_{\mu\vdash n} \chi^{\lambda/\mu}_{\sigma}\chi^{\mu}_{n} \tag{104}$$

But we earlier noted that

$$\chi^{\mu}_{n} = \begin{cases} (-1)^b & \text{if } \mu=(a+1,1^b) \\ 0 & \text{otherwise} \end{cases}$$

and hence Eq. (103) becomes

$$\chi^{\lambda}_{\sigma n} = \sum^{n-1}_{\substack{a,b=0\\ a+b+1=n}} (-1)^b\chi^{\lambda/a+1,1^b}_{\sigma} \tag{104}$$

The case of $n=1$ specialises to

$$\chi^{\lambda}_{\sigma 1} = \chi^{\lambda/1}_{\sigma} \tag{105}$$

Clearly if the class in $S(m+n)$ is $(\sigma 1^n)$ then we have by repeated application of Eq. (104) that

$$\chi^{\lambda}_{\sigma 1^n} = \chi^{\lambda/1/1\dots/1}_{\sigma} \tag{105}$$

where the skew with $\{1\}$ is repeated n times. But

$$\{1\}^{\times n} = \sum_{\mu\vdash n} f^{\{\mu\}}_n\{\mu\} \tag{106}$$

allowing us to rewrite Eq. (105) as

$$\chi^{\lambda}_{\sigma 1^n} = \sum_{\mu\vdash n} f^{\{\mu\}}_n\chi^{\lambda/\mu}_{\sigma} \tag{107}$$

and of course

$$\chi^{\lambda}_{1^n} = f^{\{\lambda\}}_n \tag{108}$$

Note the occurrence of the dimension formula in several of the above results. This suggests that further progress might be made in terms of the reduced notation and the reduced dimension formula. That will be the subject of the next lecture.

> *I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave to others the pleasure of discovery*
> René Descartes (1596 - 1650) *La Geometrie*