Symmetric Functions and the Symmetric Group 8
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> If the reader thinks he is done, now, and that this book
> has no moral to it, he is in error. The moral of it is
> this: If you are of any account, stay at home and make your
> way by faithful diligence; but if you are of "no account",
> go away from home, and then you will *have* to work, whether
> you want to or not. Thus you become a blessing to your friends by
> ceasing to be a nuisance to them
> Mark Twain *Roughing it* (1872).

8.1 Some basic branching rules

The unitary group, $U(n)$, contains many subgroups of relevance to applications in physics. Time does not permit detailed derivations, these can be found in references given earlier. Three basic branching rules can be written symbolically in terms skews of the special infinite series $S-$functions $B,\ D,\ M$ introduced earlier:-

$$U(n)\downarrow U(n-1) \qquad \{\lambda\}\downarrow\{\lambda/M\} \tag{8.1a}$$
$$U(n)\downarrow O(n) \qquad \{\lambda\}\downarrow[\lambda/D] \tag{8.1b}$$
$$U(2n)\downarrow Sp(2n) \qquad \{\lambda\}\downarrow\langle\lambda/B\rangle \tag{8.1c}$$

Recall

$$M=\{0\}+\{1\}+\{2\}+\{3\}+\{4\}+\{5\}+\{6\}+\{7\}+\{8\}+\dots \tag{8.2a}$$
$$D=\{0\}+\{2\}+\{2^2\}+\{2^3\}+\{2^4\}+\{4\}+\{42\}+\{42^2\}+\{4^2\}+\{6\}+\{62\}+\{8\}+\dots \tag{8.2b}$$
$$B=\{0\}+\{1^2\}+\{1^4\}+\{1^6\}+\{1^8\}+\{2^2\}+\{2^21^2\}+\{2^21^4\}+\{2^4\}+\{3^2\}+\{3^21^2\}+\{4^2\}+\dots \tag{8.2c}$$

Note that (8.1a-c) hold for all n the distinction comes only in the application of modification rules, if required. Thus from (8.1a) we obtain for $U(n)\downarrow U(n-1)$ the typical $n-$independent results

$$\{21\}\downarrow\{21\}+\{2\}+\{1^2\}+\{1\} \tag{8.3a}$$
$$\{31^2\}\downarrow\{31^2\}+\{31\}+\{21^2\}+\{21\}+\{1^3\}+\{1^2\} \tag{8.3b}$$
$$\{42\}\downarrow\{42\}+\{41\}+\{4\}+\{32\}+\{31\}+\{3\}+\{2^2\}+\{21\}+\{2\} \tag{8.3c}$$
$$\{32^2\}\downarrow\{32^2\}+\{321\}+\{32\}+\{2^3\}+\{2^21\}+\{2^2\} \tag{8.3d}$$

The above results hold without modification if $n\geq 4$. Actually, (8.3a) and (8.3c) hold for $n\geq 3$. For $n=2$ we would need to discard the $S-$functions involving two parts appearing on the right-hand-side of (8.3a) and (8.3c) while (8.3b) and (8.3d) would be completely null.

The branching rule for $SU(n)\downarrow SU(n-1)$ is the same as in (8.1a) with the proviso that inequivalent irreducible representations of $SU(n)$ involve at most $n-1$ non-zero parts. Thus under $SU(3)\downarrow SU(2)$ we have

$$\{21\}\downarrow 2\{1\}+\{2\}+\{0\} \tag{8.4a}$$

which dimensionally corresponds to

$$\bar{8}\downarrow 2\ \underline{2}+\underline{3}+\underline{1} \tag{8.4b}$$

8.2 Examples of $U(n)\downarrow O(n)$ branching rules

Use of (8.1b) readily leads to the typical $n-$independent results:-

$$\{21\}\downarrow[21]+[1] \tag{8.5a}$$
$$\{2^21^2\}\downarrow[2^21^2]+[21^2]+[1^2] \tag{8.5b}$$
$$\{32^21\}\downarrow[32^21]+[321]+[31]+[2^3]+[2^21^2]+[2^2]+[21^2]+[2]+[1^2] \tag{8.5c}$$

Recall that the tensor irreducible representations of $O(2k)$ and $O(2k+1)$ are labelled by partitions having at most k non-zero parts. Thus the above results would be valid without modification for $k \geq 4$. For smaller values of k the $O(n)$ modification rules must be applied. Thus for $U(6) \downarrow O(6)$ the above results would modify to

$$\{21\} \downarrow [21] + [1] \tag{8.6a}$$
$$\{2^2 1^2\} \downarrow [2^2]^* + [21^2] + [1^2] \tag{8.6b}$$
$$\{32^2 1\} \downarrow [321] + [31] + [2^3] + [2^2]^* + [2^2] + [21^2] + [2] + [1^2] \tag{8.6c}$$

since

$$[2211] \equiv [2^2]^*, \quad [32^2 1] \equiv 0, \tag{8.7}$$

The results for $SU(6) \downarrow SO(6)$ can be obtained from (8.6) by noting that under $O(2k) \downarrow SO(2k)$ we have

$$[\lambda]^\star \equiv [\lambda] \tag{8.8}$$

and if $[\lambda]$ has k non-zero parts then

$$[\lambda] \equiv [\lambda]_+ + [\lambda]_- \tag{8.9}$$

and hence for $SU(6) \downarrow SO(6)$ (8.6) becomes

$$\{21\} \downarrow [21] + [1] \tag{8.10a}$$
$$\{2^2 1^2\} \downarrow [2^2] + [21^2]_+ + [21^2]_- + [1^2] \tag{8.10b}$$
$$\{32^2 1\} \downarrow [321]_+ + [321]_- + [31] + [2^3]_+ + [2^3]_- + 2[2^2] + [21^2]_+ + [21^2]_- + [2] + [1^2] \tag{8.10c}$$

■ 8.3 Examples of $U(2n) \downarrow Sp(2n)$ branching rules

The $U(2n) \downarrow Sp(2n)$ branching rules follow from (8.1c) and use of the $B-$series leads to the typical results:-

$$\{21\} \downarrow \langle 21\rangle + \langle 1\rangle \tag{8.11a}$$
$$\{2^2 1^2\} \downarrow \langle 2^2 1^2\rangle + \langle 2^2\rangle + \langle 21^2\rangle + \langle 1^4\rangle + 2\langle 1^2\rangle + \langle 0\rangle \tag{8.11b}$$
$$\{32^2 1\} \downarrow \langle 32^2 1\rangle + \langle 321\rangle + \langle 31^3\rangle + \langle 31\rangle + \langle 2^3\rangle + \langle 2^2 1^2\rangle + \langle 2^2\rangle + 2\langle 21^2\rangle + \langle 2\rangle + \langle 1^2\rangle \tag{8.11c}$$

For $n \geq 4$ the above results are $n-$independent. For smaller values of n the modification rules must be applied to give for $SU(6) \downarrow Sp(6)$ the modified results:-

$$\{21\} \downarrow \langle 21\rangle + \langle 1\rangle \tag{8.12a}$$
$$\{2^2 1^2\} \downarrow \langle 2^2\rangle + \langle 21^2\rangle + 2\langle 1^2\rangle + \langle 0\rangle \tag{8.12b}$$
$$\{32^2 1\} \downarrow \langle 321\rangle + \langle 31\rangle + \langle 2^3\rangle + \langle 2^2\rangle + 2\langle 21^2\rangle + \langle 2\rangle + \langle 1^2\rangle \tag{8.12c}$$

In the above cases the modification rules make the terms that have four non-zero parts null. This is generally not the case.

■ 8.4 A branching rule theorem

Suppose that $\mathrm{G} \supset \mathrm{H}$ and that the vector irreducible representation 1_G branches as

$$1_\mathrm{G} \downarrow \sum_\nu \nu_\mathrm{H} \tag{8.13}$$

with ν_H not necessarily irreducible, then

$$\lambda_\mathrm{G} \downarrow (\sum_\nu \nu_\mathrm{H}) \otimes \lambda_\mathrm{G} \tag{8.14}$$

(I gave this result long ago B G Wybourne *Symmetry Principles in Atomic Spectroscopy* New York - Wiley-Interscience (1970).) This result is very useful in obtaining new branching rules as we now show.

■ 8.5 The $U(m+n) \downarrow U(m) \times U(n)$ branching rule

In this case the vector irreducible representation $\{1\}$ of $U(m+n)$ branches as

$$\{1\} \downarrow \{1\}_m + \{1\}_n \tag{8.15}$$

and from (8.14)

$$\{\lambda\}_{m+n} \downarrow (\{1\}_m + \{1\}_n) \otimes \{\lambda\} \tag{8.16a}$$
$$= \sum_\zeta (\{1\}_m \otimes \{\lambda/\zeta\}) \times (\{1\}_n \otimes \{\zeta\}) \tag{8.16b}$$
$$= \sum_\zeta \{\lambda/\zeta\}_m \times \{\zeta\}_n \tag{8.16c}$$

where in (8.16b) we have used (6.12d) and in (8.16c) used the fact that

$$\{1\} \otimes \{\mu\} \equiv \{\mu\} \tag{8.17}$$

■ 8.6 Example

Consider the irreducible representation $\{21\}$ of some group-subgroup $U(m+n) \downarrow U(m) \times U(n)$. From (8.16c) we have

$$\{21\} \downarrow \sum_\zeta \{21/\zeta\}_m \times \{\zeta\}_n \tag{8.18}$$

The sum is over all $S-$functions $\{\zeta\}$ that skew $\{21\}$ i.e. the set

$$\{\zeta\} = \{21\} + \{2\} + \{1^2\} + \{1\} + \{0\} \tag{8.19}$$

using those in (8.18) leads to the branching rule

$$\{21\} \downarrow \{21\} \times \{0\} + \{2\} \times \{1\} + \{1^2\} \times \{1\} + \{1\} \times \{2\} + \{1\} \times \{1^2\} + \{0\} \times \{21\} \tag{8.20}$$

The above results also hold for $SU(m+n) \downarrow SU(m) \times SU(n)$ with the usual partition provisos. Thus for $SU(7) \downarrow SU(4) \times SU(3)$ we can check that (8.20) is dimensionally correct by noting that

$$\underline{112} = \underline{20} \times \underline{1} + \underline{10} \times \underline{3} + \underline{6} \times \underline{3} + \underline{4} \times \underline{6} + \underline{4} \times \underline{3} + \underline{1} \times \underline{8} \tag{8.21}$$

■ 8.7 The $U(mn) \downarrow U(m) \times U(n)$ branching rule

In this case the vector irreducible representation branches as

$$\{1\}_{mn} \downarrow \{1\}_m \times \{1\}_n \tag{8.22}$$

and noting (6.12f) and (8.17) we obtain

$$\{\lambda\}_{mn} \downarrow (\{1\}_m \times \{1\}_n) \otimes \{\lambda\} \tag{8.23a}$$
$$= \sum_\rho (\{1\}_m \otimes \{\rho\}) \times (\{1\}_n \otimes \{\lambda \circ \rho\}) \tag{8.23b}$$
$$= \sum_\rho \{\rho\}_m \times \{\lambda \circ \rho\}_n \tag{8.23c}$$

Note the appearance of an inner $S-$function product $\{\lambda \circ \rho\}_n$. If $\{\lambda\} = \{1^k\}$ then recall that

$$\{1^k \circ \rho\} = \begin{cases} \{\rho\}' & \text{if } w_\rho = k \\ 0 & \text{if } w_\rho \neq k \end{cases} \tag{8.24}$$

where the partition $(\rho)'$ is the conjugate of the partition (ρ) while if $\{\lambda\} = \{k\}$ then

$$\{k \circ \rho\} = \begin{cases} \{\rho\} & \text{if } w_\rho = k \\ 0 & \text{if } w_\rho \neq k \end{cases} \tag{8.25}$$

Thus for the special case where $\{\lambda\} = \{k\}$ *or* $\{1^k\}$ we have the simplifications of (8.23), namely,

$$\{k\} \downarrow \sum_\rho \{\rho\}_m \times \{\rho\}_n \tag{8.26a}$$

$$\{1^k\} \downarrow \sum_\rho \{\rho\}_m \times \{\rho\}_n{}' \tag{8.26b}$$

where in both cases the summation over ρ is restricted to partitions (ρ) of weight k. Equation (8.26a) is appropriate to k identical bosons and (8.26b) to k identical fermions.

■ 8.8 Classification of the states of the d^N electron configurations

A $d-$orbital has spin $S = \frac{1}{2}$ and orbital angular momentum $L = 2$. Thus there are 10 spin-orbital states that can be regarded as forming a basis for the vector irreducible representation $\{1\}$ of the special unitary group $SU(10)$. the spin and orbital parts of the wave function will span the direct product subgroup $SU(2) \times SU(5)$ subgroup of $SU(10)$. If we have N electrons in equivalent $d-$orbitals (same principal quantum number n) to satisfy the Pauli exclusion principle their wavefunction must be totally antisymmetric with respect to the spin-orbital quantum numbers. This will be the case if they span the irreducible representation $\{1^N\}$ of $SU(10)$. Thus to determine the total spin S and orbital L quantum numbers we need to first use (8.26b) to determine the

$$SU(10) \downarrow SU(2)^S \times SU(5)^L \tag{8.27}$$

branching rules for the $\{1^N\}$ irreducible representation of $SU(10)$. Inequivalent irreducible representations of $SU(2)$ are labelled by single non-negative integers while irreducible representations of the type $\{p, q\}$ are associated with the equivalence

$$\{p, q\} \equiv \{p - q\} \qquad p \geq q \geq 0 \tag{8.28}$$

An irreducible representation $\{p\}$ of $SU(2)$ will be of dimension

$$dimension(\{p\}) = p + 1 \tag{8.29}$$

and spin

$$spin(\{p\}) = S = \frac{p}{2} \tag{8.30}$$

Thus from (8.26b) we have under $SU(10) \downarrow SU(2) \times SU(5)$

$$\{1^N\} \downarrow \sum_{\substack{\sigma_1 \geq \sigma_2 \geq 0 \\ \sigma_1 + \sigma_2 = N}} \{\sigma_1, \sigma_2\} \times \{\sigma_1, \sigma_2\}' \tag{8.31a}$$

$$= \sum_{\substack{\sigma_1 \geq \sigma_2 \geq 0 \\ \sigma_1 + \sigma_2 = N}} \{\sigma_1 - \sigma_2\} \times \{2^{\sigma_2} 1^{\sigma_1 - \sigma_2}\} \tag{8.31b}$$

where we have made use of the fact that $SU(2)$ limits (σ) in (8.26b) to two row partitions and the conjugate partition must then be a two column partition. Below we give a table of the branching rules for $N = 0, 1, \ldots, 10$.

Table 8.1 Branching rules for the $\{1^N\}$ irreducible representations under $SU(10) \downarrow SU(2) \times SU(5)$

Dim	$SU(10) \downarrow$	$SU(2) \times SU(5)$
1	$\{0\}$	$\{0\} \times \{0\}$
10	$\{1\}$	$\{1\} \times \{1\}$
45	$\{1^2\}$	$\{2\} \times \{1^2\} + \{0\} \times \{2\}$
120	$\{1^3\}$	$\{3\} \times \{1^3\} + \{1\} \times \{21\}$
210	$\{1^4\}$	$\{4\} \times \{1^4\} + \{2\} \times \{21^2\} + \{0\} \times \{2^2\}$
252	$\{1^5\}$	$\{5\} \times \{0\} + \{3\} \times \{21^3\} + \{1\} \times \{2^21\}$
210	$\{1^6\}$	$\{4\} \times \{1\} + \{2\} \times \{2^21^2\} + \{0\} \times \{2^3\}$
120	$\{1^7\}$	$\{3\} \times \{1^2\} + \{1\} \times \{2^31\}$
45	$\{1^8\}$	$\{2\} \times \{1^3\} + \{0\} \times \{2^4\}$
10	$\{1^9\}$	$\{1\} \times \{1^4\}$
1	$\{1^{10}\}$	$\{0\} \times \{0\}$

(8.32)

Notice that the partitions labelling the irreducible representations of $SU(5)$ involve at most 4 non-zero integers. The single integers labelling the irreducible representations of $SU(2)$ are twice the value of the spin S and all the states belonging to the associated $SU(5)$ irreducible representation have that spin quantum number. Further, notice that the dimensions of the $SU(10)$ irreducible representations $\{1^N\}$ and $\{1^{10-N}\}$ are equal and that the $SU(2)$ spins are the same with each $SU(5)$ being the contragredient partner. This is the familiar particle-hole symmetry manifesting itself.

To proceed further we need to branch the $SU(5)$ irreducible representations into those of its subgroup $SO(5)$ using (8.1b). We note that the branchings for contragredient partners are identical so that it suffices to consider just those irreducible representations of $SU(5)$ that occur in Table 8.1 for $N \leq 5$.

Table 8.2 Branching rules for $SU(5) \downarrow SO(5)$.

Dim	$SU(5) \downarrow$	$SO(5)$
1	$\{0\}$	$[0]$
5	$\{1\}$	$[1]$
10	$\{1^2\}$	$[1^2]$
15	$\{2\}$	$[2]+[0]$
10	$\{1^3\}$	$[1^2]$
40	$\{21\}$	$[21]+[1]$
5	$\{1^4\}$	$[1]$
45	$\{21^2]$	$[21]+[1^2]$
50	$\{2^2\}$	$[2^2]+[2]+[0]$
1	$\{1^5\}$	$[0]$
24	$\{21^3\}$	$[2]+[1^2]$
75	$\{2^2 1\}$	$[2^2]+[21]+[1]$ (8.33)

Finally, to complete the classification of the states of d^N we need the branching rules for $SO(5) \downarrow SO(3)$. These can be found by use of the branching rule theorem of (8.14). Since under $SO(5) \downarrow SO(3)$

$$[1] \downarrow [2] \tag{8.34}$$

we have from (8.14) that

$$[\lambda] \downarrow [2] \otimes [\lambda] \tag{8.35a}$$

$$= [((\{2\} - \{0\}) \otimes \{\lambda/C\})/D] \tag{8.35b}$$

In going from (8.35a) to (8.35b) we have made use of the possibility of inverting the $SU(5) \downarrow SO(5)$ branching rule by use of the $C-$series of $S-$functions that is the inverse of the $D-$series. Thus in going from (8.35a) to (8.35b) we have replaced [2] by $\{2/C\} = \{2\} - \{0\}$ and $[\lambda]$ by $\{\lambda/C\}$ and then evaluated the plethysms as for $S-$functions and then skew the resultant list of $S-$functions with the $D-$series and applied the $SO(3)$ modification rules to reduce everything to a list of single part $SO(3)$ labels and given them their standard spectroscopic angular momentum labels L where the correspondence is

$$[0]S, [1]P, [2]D, [3]F, [4]G, [5]H, [6]I, [7]K, [8]L, [9]M, [10]N, [11]O, [12]Q, \ldots \tag{8.36}$$

Table 8.3 Some $SO(5) \downarrow SO(3)$ branching rules.

Dim	$SO(5) \downarrow$	$SO(3)$
1	$[0]$	S
5	$[1]$	D
10	$[1^2]$	$P+F$
14	$[2]$	$D+G$
35	$[21]$	$P+D+F+G+H$
35	$[2^2]$	$S+D+F+G+I$ (8.37)

Thus we have all the components to classify the states of the d^N electron configurations using the group-subgroup scheme

$$SU(10) \supset SU(2)^S \times (SU(5) \supset SO(5) \supset SO(3)^L) \tag{8.38}$$

Table 8.4 Group classification of the states of the d^N electron configurations.

d^N	$SU(2) \times SU(5)$	$SO(5)$	$SO(3)$	^{2S+1}L
d^0	$\{0\} \times \{0\}$	$[0]$	$[0]$	^1S
d^1	$\{1\} \times \{1\}$	$[1]$	$[1]$	^2D
d^2	$\{2\} \times \{1^2\}$	$[1^2]$	$[1] + [3]$	^3PF
	$\{1^2\} \times \{2\}$	$[2]$	$[2] + [4]$	^1DG
		$[0]$	$[0]$	^1S
d^3	$\{3\} \times \{1^3\}$	$[1^2]$	$[1] + [3]$	^4PF
	$\{1\} \times \{21\}$	$[21]$	$[1] + [2] + [3] + [4] + [5]$	^2PDFGH
		$[1]$	$[2]$	^2D
d^4	$\{4\} \times \{1^4\}$	$[1]$	$[2]$	^5D
	$\{2\} \times \{21^2\}$	$[21]$	$[1] + [2] + [3] + [4] + [5]$	^3PDFGH
		$[1]$	$[2]$	^3D
	$\{0\} \times \{2^2\}$	$[2^2]$	$[0] + [2] + [3] + [4] + [6]$	^1SDFGI
		$[2]$	$[2] + [4]$	^1DG
		$[0]$	$[0]$	^1S
d^5	$\{5\} \times \{0\}$	$[0]$	$[0]$	^6S
	$\{3\} \times \{21^3\}$	$[2]$	$[0] + [2]$	^4SD
		$[1^2$	$[1] + [3]$	^4PF
	$\{1\} \times \{2^21\}$	$[2^2]$	$[0] + [2] + [3] + [4] + [6]$	^2SDFGI
		$[21]$	$[1] + [2] + [3] + [4] + [5]$	^2PDFGH
		$[1]$	$[2]$	^2D

(8.39)

Note that every state has a distinct set of labels. This would not be the case if we had simply enumerated the ^{2S+1}L states.

■ 8.9 Seniority classification of the states of d^N

We could have used several other possible subgroup structures embedded in $SU(10)$ to classify the states of d^N. Let us consider the group-subgroup structure

$$SU(10) \downarrow Sp(10) \downarrow SU(2) \times (SO(5) \downarrow SO(3)) \tag{8.40}$$

The $SU(10) \downarrow Sp(10)$ branching rules follow from (8.1c) to give the results in Table 8.5.

Table 8.5 Some $SU(10) \downarrow Sp(10)$ branching rules.

$SU(10) \downarrow$	$Sp(10)$
$\{0\}$	$\langle 0 \rangle$
$\{1\}$	$\langle 1 \rangle$
$\{1^2\}$	$\langle 1^2 \rangle + \langle 0 \rangle$
$\{1^3\}$	$\langle 1^3 \rangle + \langle 1 \rangle$
$\{1^4\}$	$\langle 1^4 \rangle + \langle 1^2 \rangle + \langle 0 \rangle$
$\{1^5\}$	$\langle 1^5 \rangle + \langle 1^3 \rangle + \langle 1 \rangle$

(8.41)

The branching rules for $Sp(2k) \downarrow SU(2) \times SO(2k+1)$ follow from the branching rule theorem to give generally

$$\langle \lambda \rangle \downarrow \sum_{\sigma} \{\lambda / A \circ \sigma\} \times [\sigma / D] \tag{8.42}$$

The evaluation may be readily done in SCHUR either using the branching rule 11 or writing the simple function

```
gr sp10
enter rv1
dim[rv1]
gr2su2so5
rule[rv1*0]sk1with a
rule last sum i1eq2
rule last sk2with d
supout false
std last
dim last
stop
```

Running the function gives for example

```
DP>
fn1
Group is Sp(10)
enter rv1
1111
Dimension = 165
Groups are   SU(2) * SO(5)
      {4}[1] + {2}[21] + {0}[2^2 ]
Dimension = 165
DP>
```

Thus we obtain the $Sp(10) \downarrow SU(2) \times SO(5)$ branching rules given in Table 8.6.

Table 8.6 Some $Sp(10) \downarrow SU(2) \times SO(5)$ branching rules.

Dim	$Sp(10) \downarrow$	$SU(2) \times SO(5)$
1	$\langle 0 \rangle$	$\{0\} \times [0]$
10	$\langle 1 \rangle$	$\{1\} \times [1]$
44	$\langle 1^2 \rangle$	$\{2\} \times [1^2] + \{0\} \times [2]$
110	$\langle 1^3 \rangle$	$\{3\} \times [1^2] + \{1\} \times [21]$
165	$\langle 1^4 \rangle$	$\{4\} \times [1] + \{2\} \times [21] + \{0\} \times [2^2]$
132	$\langle 1^5 \rangle$	$\{5\} \times [0] + \{3\} \times [2] + \{1\} \times [2^2]$

(8.43)

Combining the results of Tables 8.5 and 8.6 with that of 8.3 gives the classification of the d^N states shown in Table 8.7.

Table 8.7 The symplectic classification of the states of the d^N electron configurations.

d^N	$SU(10)\downarrow$	$Sp(10)\downarrow$	$SU(2)\times SO(5)\downarrow$	^{2S+1}L
d^0	$\{0\}$	$\langle 0\rangle$	$\{0\}\times[0]$	^1S
d^1	$\{1\}$	$\langle 1\rangle$	$\{1\}\times[1]$	^2D
d^2	$\{1^2\}$	$\langle 1^2\rangle$	$\{2\}\times[1^2]$	^3PF
			$\{0\}\times[2]$	^1DG
		$\langle 0\rangle$	$\{0\}\times[0]$	^1S
d^3	$\{1^3\}$	$\langle 1^3\rangle$	$\{3\}\times[1^2]$	^4PF
			$\{1\}\times[21]$	^2PDFGH
		$\langle 1\rangle$	$\{1\}\times[1]$	^2D
d^4	$\{1^4\}$	$\langle 1^4\rangle$	$\{4\}\times[1]$	^5D
			$\{2\}\times[21]$	^3PDFGH
			$\{0\}\times[2^2]$	^1SDFGI
		$\langle 1^2\rangle$	$\{2\}\times[1^2]$	^3PF
			$\{0\}\times[2]$	^1DG
		$\langle 0\rangle$	$\{0\}\times[0]$	^1S
d^5	$\{1^5\}$	$\langle 1^5\rangle$	$\{5\}\times[0]$	^6D
			$\{3\}\times[2]$	^4DG
			$\{1\}\times[2^2]$	^2SDFGI
		$\langle 1^3\rangle$	$\{3\}\times[1^2]$	^4PF
			$\{1\}\times[21]$	^2PDFGH
		$\langle 1\rangle$	$\{1\}\times[1]$	^2D (8.44)

A careful inspection of the above table reveals a striking property, known as *seniority*.In d^2 we note that the ^1S state has the same $Sp(10)$ label as does the corresponding state for d^0. It is as if in going from d^0 to d^2 two $d-$electrons have paired to produce an angular momentum state $L = 0$. Let the integer v be the value of N for which the $Sp(10)$ irreducible representation $\langle 1^N\rangle$ first occurs then $\frac{N-v}{2}$ is the number of pairs of $d-$electrons coupled to zero angular momentum in forming the $N-$particle state. For example $\langle 1^2\rangle$ occurs in d^2, d^4 and those states are assigned seniority $v = 2$. Seniority is a useful concept in calculating matrix elements in atomic physics and of much greater usefulness in nuclear shell calculations where strong pairing interactions occur. In nuclei states of lowest energy have lowest seniority whereas in atomic shells one has the opposite situation.