Symmetric Functions and the Symmetric Group 6

B. G. Wybourne

This is why I value that little phrase "I don't know" so highly. It's small, but it flies on mighty wings. It expands our lives to include the spaces within us as well as those outer expanses in which our tiny Earth hangs suspended. If Isaac Newton had never said to himself "I don't know," the apples in his little orchard might have dropped to the ground like hailstones and at best he would have stooped to pick them up and gobble them with gusto. Had my compatriot Marie Sklodowska-Curie never said to herself "I don't know", she probably would have wound up teaching chemistry at some private high school for young ladies from good families, and would have ended her days performing this otherwise perfectly respectable job. But she kept on saying "I don't know," and these words led her, not just once but twice, to Stockholm, where restless, questing spirits are occasionally rewarded with the Nobel Prize.

WISLAWA SZYMBORSKA (Nobel Lecture 1996)

6.1 Plethysm of $S-$functions

The plethysm of $S-$functions is a property that has many important applications in symmetry aspects of many-body problems in physics and grew out of the mathematical theory of invariants though nowadays forms an integral part of combinatorial mathematics. There is a close connection between the plethysm of $S-$functions and branching rules.

Let $\Lambda^n = \Lambda^n(x_1, \ldots, x_N)$ denote the space of homogeneous symmetric polynomials of degree n.Then given symmetric polynomials with integer coefficients

$$\mathrm{P} \in \Lambda^n \qquad \text{and} \qquad \mathrm{Q} \in \Lambda^m$$

then

$$\mathrm{P}[\mathrm{Q}] \qquad \text{is a symmetric polynomial in} \qquad \Lambda^{mn} \tag{6.1}$$

In this sense a plethysm can be seen as a substitution process. As a simple example consider the power sum symmetric functions

$$p_n = \sum_i x_i^n \qquad \text{and} \qquad p_m = \sum_i x_i^m$$

then

$$p_n[p_m] = p_m[p_n] = p_{mn} \tag{6.2}$$

Likewise

$$p_n[e_m] = e_m[p_n] = m_n^m \tag{6.3}$$

and

$$p_n[m_\mu] = m_\mu[p_n] = m_{\mu\cdot n} \tag{6.4}$$

where $\mu \cdot n$ signifies that each part of μ is multiplied by the integer n.

The above examples are all commutative which is not the general case. In general the $S-$function content of $s_\lambda[s_\mu]$ is not the same as that of $s_\mu[s_\lambda]$.

As an example of $S-$function plethysm consider the evaluation of $s_2[s_{1^2}](x_1, \ldots, x_4)$. We express $s_{1^2}(x_1, \ldots, x_4)$ as a sum of monomials,

$$s_{1^2}(x_1, \ldots, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \tag{6.5}$$

Now regard s_2 as a function in as many monomials as in (6.5) i.e.

$$s_2[s_{1^2}](x_1, \ldots, x_4) = s_2(x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4) \tag{6.6}$$

Very tediously, the right-hand-side of (6.6) may be expanded as a sum of monomials which in turn may be expressed in terms of $S-$functions to yield, finally

$$s_2[s_{1^2}](\mathbf{x}) = s_{2^2}(\mathbf{x}) + s_4(\mathbf{x}) \tag{6.7}$$

Noting that

$$s_2(x_1, \ldots, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \tag{6.8}$$

We have

$$s_{1^2}[s_2](x_1,\ldots,x_4) = s_{1^2}(x_1^2+x_2^2+x_3^2+x_4^2+x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4) \qquad (6.9)$$

which may be expanded as a sum of monomials and then into $S-$functions to give

$$s_{1^2}[s_2](\mathbf{x}) = s_{31}(\mathbf{x}) \qquad (6.10)$$

which is different from (6.7).

While the above examples have involved just four variables the results actually hold for any number of variables $n \geq 4$.

■ **Exercise**

1. Show that $s_{1^2}[s_{1^2}](\mathbf{x}) = s_{21^2}(\mathbf{x})$

■ **6.2 Plethysm Notation**

The plethysm of $S-$functions was introduced by D E Littlewood in terms of invariant matrices and who used the notation $\{\lambda\}\otimes\{\mu\}$. This notation is used almost universally by physicists whereas the corresponding plethysm viewed as an $S-$function substitution is almost universally written by combinatorists as $s_\mu[s_\lambda]$ the corrspondence between the two notations being

$$\{\lambda\}\otimes\{\mu\} \equiv s_\mu[s_\lambda] \qquad (6.11)$$

It much that follows we shall use the physicists notation.

■ **6.3 The algebra of plethysms**

The algebra of plethysms is governed by the rules

$$A\otimes(B\pm C) = A\otimes B \pm A\otimes C \qquad (6.12a)$$

$$A\otimes(BC) = (A\otimes B)(A\otimes C) \qquad (6.12b)$$

$$A\otimes(B\otimes C) = (A\otimes B)\otimes C \qquad (6.12c)$$

$$(A+B)\otimes\{\mu\} = \sum_\zeta (A\otimes\{\mu/\zeta\})(B\otimes\{\zeta\}) \qquad (6.12d)$$

$$(A-B)\otimes\{\mu\} = \sum_\zeta (-1)^{w_\zeta}(A\otimes\{mu/\zeta\})(B\otimes\{\tilde{\zeta}\}) \qquad (6.12e)$$

$$(AB)\otimes\{\mu\} = \sum_\rho (A\otimes\{\rho\})(B\otimes\{\mu\circ\rho\}) \qquad (6.12f)$$

Note that (6.12c) shows the associativity of the plethysm operation and that in (6.12f) the $\circ$ signifies an inner product of $S-$functions so that $\{\mu\}\circ\{\rho\}$ is the Kronecker product of irreducible representations of S_m labelled μ and ρ which are both partitions of m. In (6.12e) w_ζ is the weight of the partition (ζ) and the partition $(\tilde{\zeta})$ is conjugate to (ζ).

■ **6.4 Plethysm and $S-$function series**

Later we shall show that plethysm gives a powerful tool for developing symbolic representations of branching rules for going from the representation of a group $\mathcal{G}$ to those of a subgroup $\mathcal{H}$. However, we must first consider plethysm and $S-$function series. The basic ideas are developed in

1. M Yang and B G Wybourne, *New $S-$function series and non-compact Lie groups* J Phys A:Math.Gen. **19** 3513 (1986)
2. R C King, B G Wybourne and M Yang, *Slinkies and the $S-$function content of certain generating functions* J Phys A:Math.Gen. **22** 4519 (1989)

Consider the infinite $S-$function series

$$L(x) = \prod_{i=1}^{\infty}(1-x_i) \qquad (6.13a)$$

$$= \sum_{m=0}^{\infty}(-1)^m\{1^m\} \qquad (6.13b)$$

The inverse L^{-1} is

$$L^{-1} = \left(\prod_{i=1}^{\infty}(1-x_i)\right)^{-1} = \prod_{m=0}^{\infty}\{m\} = M \tag{6.14}$$

Let us define the adjoint series $L^{\dagger}$ as the conjugate ($\tilde{}$) inverse or the inverse conjugate of L:

$$L^{\dagger} = (\tilde{L})^{-1} = \tilde{L}^{-1} \tag{6.15}$$

leading to

$$L^{\dagger} = \prod_{i=1}^{\infty}(1+x_i) = \sum_{m=0}^{\infty}\{1^m\} = Q \tag{6.16}$$

Note that taking the adjoint ($\dagger$) is equivalent to the substitution

$$x_i \to -x_i \tag{6.17}$$

in $L(x_i)$, which can be viewed as a plethysm:

$$L^{\dagger} = L(-x_i) = (-\{1\}) \otimes L \tag{6.18}$$

The conjugate of L is also the inverse of $L^{\dagger}$ and hence

$$\tilde{L} = (L^{\dagger})^{-1} = \left(\prod_{i=1}^{\infty}(1+x_i)\right)^{-1} = \prod_{m=0}^{\infty}(-1)^m\{m\} = P \tag{6.19}$$

We thus have four infinite $S-$function series L, M, P, Q related by the four properties, identity (I), conjugation ($\tilde{}$), inverse (-1) and adjoint ($\dagger$) which form a discrete four-element group with the Cayley table

	I	$\tilde{}$	-1	$\dagger$
I	I	$\tilde{}$	-1	$\dagger$
$\tilde{}$	$\tilde{}$	I	$\dagger$	-1
-1	-1	$\dagger$	I	$\tilde{}$
$\dagger$	$\dagger$	-1	$\tilde{}$	I

Having obtained the four L type series we can obtain further series by simple substitution into the L series. Thus under the substitution

$$x_i \to x_i x_j \quad (i<j) \tag{6.20}$$

we obtain

$$L(x_i x_j) = \prod_{(i<j)}^{\infty}(1-x_i x_j) \tag{6.21a}$$

$$= \{1^2\} \otimes L \tag{6.21b}$$

$$= \sum_{\alpha}(-1)^{w_\alpha}\{\alpha\} = A \tag{6.21c}$$

where in the Frobenius notation

$$(\alpha) = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_r \\ \alpha_1+1 & \alpha_2+1 & \dots & \alpha_r+1 \end{pmatrix} \tag{6.22}$$

Continuing we could construct four series $A,\ A^{-1},\ \tilde{A},\ A^{\dagger}$.

The substitution

$$x_i \to x_i^2 \tag{6.23}$$

leads to

$$L(x_i^2) = \prod_{i=1}^{\infty}(1 - x_i^2) \tag{6.24a}$$

$$= (\{2\} - \{1^2\}) \otimes L \tag{6.24b}$$

$$= \sum_{p,q=0}^{\infty}(-1)^p\{p + 2q, p\} = V \tag{6.24c}$$

■ 6.5 Why are infinite $S-$function series important?

We noted earlier that $S-$functions can be related to the characters of the unitary groups $U(n)$ and the $S-$function multiplication via the Littlewood-Richardson rule corresponds to the resolution of Kronecker products of irreducible representations in $U(n)$. We also noted that a given irreducible representation of $U(n)$, say $\{\lambda\}$ becomes reducible under the group-subgroup restriction $U(n) \to U(n-1)$ such that

$$\{\lambda\} \to \{\lambda/M\} \tag{6.25}$$

where M is the infinite $S-$function series

$$M = \sum_{m=0}^{\infty}\{m\} \tag{6.26}$$

The number of terms is rendered finite by the occurrence of the M series as a $S-$function skew. The irreducible representations of $U(n)$ are all finite dimensional so the occurrence of the $S-$function series as skews is to be expected. However, there are, so-called non-compact groups whose non-trivial unitary representations are infinite-dimensional. In those cases the characters may be represented in terms of infinite $S-$function series and upon restriction to compact subgroups the branching rules involving an infinite number of representations of the compact subgroup and the $S-$function series appear in the numerator rather than as skews. Likewise whereas for compact groups, like $U(n)$, the Kronecker products involve a finite number of terms for the noncompact groups a Kronecker product of a pair of infinite-dimensional irreducible representations will usually involve an infinite number of infinite dimensional unitary irreducible representations. We shall not explore non-compact groups in any detail here. The interested reader may explore some of the references below.

1. D. J. Rowe, B. G. Wybourne and P. H. Butler, *Unitary Representations, Branching Rules and Matrix Elements for the Non-Compact Symplectic Groups,* J Phys A:Math.Gen. **18**, 939-953 (1985)
2. R. C. King and B. G. Wybourne, *Holomorphic Discrete Series and Harmonic Series Unitary Irreducible Representations of Non-Compact Lie Groups: $Sp(2n,\Re)$, $U(p,q)$ and $SO^*(2n)$,* J Phys A:Math.Gen. **18**, 3113-3139 (1985)
3. B. G. Wybourne, *The representation space of the nuclear symplectic $Sp(6,\Re)$ shell model* J Phys A:Math.Gen. **25**, 4389-4398 (1992)
4. K. Grudzinski and B. G. Wybourne, *Plethysm for the noncompact group $Sp(2n,\Re)$ and new $S-$function identities* J Phys A:Math.Gen. **29**, 6631-6641 (1996).
5. Jean-Yves Thibon, Frederic Toumazet and Brian G Wybourne, *Products and plethysms for the fundamental harmonic series representations of $U(p,q)$*, J Phys A:Math.Gen. **30**, 4851-6 (1997)
6. Jean-Yves Thibon, Frederic Toumazet and Brian G Wybourne, *Symmetrised squares and cubes of the fundamental unirreps of $Sp(2n,\Re)$*, J Phys A:Math.Gen. **31**, 1073-86 (1998)
7. R C King and B G Wybourne, Products and symmetrised powers of irreducible representations of $Sp(2n,\Re)$ and their associates, **31**,6669-6689 (1998)
8. R C King, F. Toumazet and B G Wybourne, *Products and symmetrised powers of irreducible representations of $SO^*(2n)$*, J Phys A:Math.Gen. **31**, 6691-6705 (1998)
9. R C King and B G Wybourne, *Analogies between finite-dimensional irreps of $SO(2n)$ and infinite-dimensional irreps of $Sp(2n,\Re)$ Part I: Characters and products* , J.Math.Phys. **41**, 5002-19 (2000)

10. R C King and B G Wybourne, *Analogies between finite-dimensional irreps of $SO(2n)$ and infinite-dimensional irreps of $Sp(2n,\Re)$ Part II:Plethysms*, J.Math.Phys. **41**,5656-90 (2000)

■ 6.7 Regular matrix groups

Consider square $n \times n$ matrices A such that

1. The unit element is the $n \times n$ identity matrix,

$$\mathrm{I} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & 1 \end{pmatrix} \tag{27}$$

2. The existence of an inverse element, A^{-1}, is assured by restriction to non-singular matrices such that

$$\det |A| \neq 0 \tag{6.28}$$

3. The laws of matrix multiplication are such that the associative law of multiplication is satisfied.
4. The set of matrices is such that closure is assured.

If the above four properties are satisfied then the set of matrices will form a group. Groups involving regular matrices may be finite or infinite, be discrete or continuous, and have real ($\Re$) or complex (C) elements. The variables in the real space $\Re^n$ will be designated $\mathbf{x} \equiv (x_1, \ldots, x_n)$ and in the complex space C^n as $\mathbf{z} \equiv (z_1, \ldots, z_n)$. A regular matrix of degree n acting in ($\Re^n$) or in C^n will produce transformations $\mathbf{x} \to \mathbf{x}'$ or $\mathbf{z} \to \mathbf{z}'$. In problems in physics we are frewquently interested classes of transformations that leave invariant some functional form of $\mathbf{x}$ or $\mathbf{z}$.

■ 6.8 Continuous matrix groups

Consider a group whose elements comprise all regular nonsingular real matrices of degree 2,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{6.29}$$

Apart from the constraint

$$a_{11}a_{22} \neq a_{12}a_{21} \tag{6.30}$$

that follows from (6.28), the range of the elements of the matrix is unrestricted and we can parameterise the matrix elements a_{ij} as

$$a_{ij} = \delta_{ij} + \alpha_{ij} \tag{6.31}$$

If all $\alpha_{ij} = 0$ we simply obtain the identity matrix

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.32}$$

We can treat the α_{ij} as real independent parameters and generate all the elements of the group by a continuous variation of the α_{ij}. The range of the parameters is unbounded and limited only to the extent demanded by (6.30). Any element of the group can be designated by giving its associated values of the parameters α_{ij}.

■ Exercises

1. Show that the transformations produced by the matrices

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad (0 \leq \theta < 2\pi) \tag{6.33}$$

acting in $\Re^2$ leave invariant the form $x_1^2 + x_2^2$.

2. Show that the transformations produced by the matrices

$$\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \tag{6.34}$$

leave invariant the real quadratic form $x_1^2 - x_2^2$.

■ 6.9 Matrix groups - Examples

The general linear groups $GL(n, C)$ and $GL(n, \Re)$

The *complex general linear group* $GL(n, \mathrm{C})$ is the group of regular invertible complex matrices of degree n. A particular matrix is characterised by its n^2 elements with each element containg a real and an imaginary part. The continuous variation of the $2n^2$ parts (i.e. n^2 real and n^2 complex parts) will generate the entire group and hence the group is of dimension $2n^2$ and may be characterised by $2n^2$ real parameters.

If the elements of $GL(n, \mathrm{C})$ are restricted to real values only, then

$$GL(n, \mathrm{C}) \supset GL(n, \Re) \tag{6.35}$$

The special linear groups $SL(n, C)$ and $SL(n, \Re)$

These groups occur as subgroups of $GL(n, \mathrm{C})$ and $GL(n, \Re)$ respectively when the requirement that the determinant of their matrices be of determinant $+1$. Clearly, $SL(n, \mathrm{C})$ becomes a $2(n^2 - 1)$ parameter group and $SL(n, \Re)$ a $(n^2 - 1)$ parameter group and

$$GL(n, \mathrm{C}) \supset SL(n, \mathrm{C}) \supset SL(n, \Re) \tag{6.36}$$

The special linear groups are often referred to as *special unimodular groups.*

The unitary groups

The unitary matrices A of degree n form the elements of the n^2-parameter *unitary group* $U(n)$ that leaves invariant the Hermitian form

$$\sum i = 1^n z_i z_j^* \tag{6.37}$$

Since the unitarity of the matrices A requires that

$$A^\dagger A = \mathrm{I} \tag{6.38}$$

the range of matrix elements a_{ij} is restricted by the requirement that

$$\sum_t a_{it} a_{tj}^* = \delta_{ij} \tag{6.39}$$

and hence $|a_{ij}|^2 \leq 1$. In this case the parameter domain is bounded and $U(n)$ is an example of a *compact* group.

The special unitary group $SU(n)$

If we limit our attention to unitary matrices of determinant $+1$ we obtain the $(n^2 - 1)$-parameter special unitary group $SU(n)$.

The orthogonal groups

The group of *complex* orthogonal matrices of degree n form a $n(n-1)$- parameter group $O(n, \mathrm{C})$. Since ${}^tAA = \mathrm{I}$ we have $|A| = \pm 1$ and thus the group decomposes into two disconnected pieces and we cannot pass continuously from one piece to the other. The orthogonal matrices of determinant $+1$ form a subgroup of $O(n, \mathrm{C})$, the $n(n-1)$-parameter *special complex orthogonal group* $SO(n, \mathrm{C})$ whose matrices leave invariant the complex quadratic form

$$\sum_{i=1}^n z_i^2 \tag{6.40}$$

The special real orthogonal groups $O(n, \Re)$ and $SO(n, \Re)$

The set of real orthogonal matrices of degree n forms the $n(n-1)/2$- parameter real orthogonal group $O(n, \Re)$ while the set of real orthogonal matrices of determinant $+1$ form the real special orthogonal group $SO(n, \Re)$. Again $O(n, \Re)$ contains two disconnected pieces with $SO(n, \Re)$ occurring as a subgroup. The real special orthogonal group holds invariant the real quadratic form

$$\sum_{i=1}^n x_i^2 \tag{6.41}$$

The Symplectic groups $Sp(n, C)$ and $Sp(n, \Re)$

The symplectic group $Sp(n, \mathrm{C})$ is the $2n(2n+1)$-parameter group of regular complex matrices which hold invariant the non-degenerate skew-symmetric bilinear form

$$\sum_{i=1}^{n}(x_i y_i' - x_i' y_i) \tag{6.42}$$

of two vectors $\mathbf{x} \equiv (x_1, \ldots, x_n, x_i', \ldots, x_n')$ and $\mathbf{y} \equiv (y_1, \ldots, y_n, y_i', \ldots, y_n')$. $GL(n, \mathrm{C}) \supset Sp(2n, \mathrm{C})$ and the matrices need not be unitary. Restriction to real matrices gives the $n(2n+1)$-parameter group $Sp(2n, \Re)$.

The symplectic group $sp(2n) = U(2n) \cup Sp(2n, C)$ is known as the *unitary symplectic group*. This group, like $Sp(2n, \Re)$, is a $n(2n+1)$ -parameter group. The symplectic groups occur only in even-dimensional spaces and find applications in many areas of physics.