Symmetric Functions and the Symmetric Group 3

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For every complex question there is a simple answer
and it's wrong.
H. L. Mencken

- 3.1 Semistandard numbering and Young tableaux
- Many different prescriptions can be given for injecting numbers into the boxes of a frame.
- The standard numbering is intimately associated with the symmetric group $\mathcal{S}_n$.
- Another important numbering prescription is that of *semistandard* numbering where now numbers 1, 2, ..., d are injected into the boxes of a frame F^λ such that:

  i. Numbers are non-decreasing across a row going from left to right.

  ii. Numbers are positively increasing in columns from top to bottom.

- The first condition permits repetitions of integers.

Using the numbers 1, 2, 3 in the frame $F^{2\,1}$ we obtain the 8 tableaux

$$\begin{array}{ccccc} \begin{array}{cc} 1 & 1 \\ 2 & \end{array} & \begin{array}{cc} 1 & 1 \\ 3 & \end{array} & \begin{array}{cc} 1 & 2 \\ 2 & \end{array} & & \\ \begin{array}{cc} 1 & 2 \\ 3 & \end{array} & \begin{array}{cc} 1 & 3 \\ 2 & \end{array} & \begin{array}{cc} 1 & 3 \\ 3 & \end{array} & \begin{array}{cc} 2 & 2 \\ 3 & \end{array} & \begin{array}{cc} 2 & 3 \\ 3 & \end{array} \end{array} \tag{3.1}$$

Had we chosen $d = 2$ we would have obtained just two tableaux while $d = 4$ yields twenty tableaux. In general, for a frame F^λ a semistandard numbering using the integers 1, 2, ..., d leads to

$$f_d^\lambda = \frac{G_d^\lambda}{H_\lambda} \tag{3.2}$$

where H_λ is the product of the hook lengths h_{ij} of the frame and

$$G_d^\lambda = \prod_{(i,j)\in\lambda} (d+i-j) \tag{3.3}$$

Thus for $d = 5$ and $\lambda = (4\,2\,1)$ we have $H_{(4\,2\,1)} = 144$ and $G_5^{\{4\,2\,1\}} = 100800$ from which we deduce that

$$f_5^{\{4\,2\,1\}} = 700$$

which is the dimension of the irreducible representation $\{4\,2\,1\}$ of the general linear group $GL(5)$.

- In general, f_d^λ is the dimension of the irreducible representation $\{\lambda\}$ of $GL(d)$. Since the representations of $GL(d)$ labelled by partitions λ remain irreducible under restriction to the unitary group $U(d)$ Eq.(3.3) is valid for computing the dimensions of the irreducible representations of the unitary group $U(d)$.
- The same rules for a semistandard numbering may be applied to the skew frames $F^{\lambda/\mu}$. Thus for $F^{542/21}$ an allowed semistandard numbering using just the integers 1 and 2 would be

$$\begin{array}{ccccc} & & 1 & 1 & 1 \\ & 1 & 2 & 2 & \\ 1 & 2 & & & \end{array}$$

- Note that our semistandard numbering yields what in the mathematical literature are commonly referred to as *semistandard* Young tableaux. Other numberings are possible and have been developed for all the classical Lie algebras.

■ Exercises

3.1 Draw the frames $F^{2^2/1}$, $F^{43^21/421^2}$, and $F^{321/21}$.

3.2 Use the integers 1, 2, 3 to construct the complete set of semistandard tableaux for the frame $F^{43^21/421^2}$ and show that the same number of tableaux arise for the frame F^{21}.

3.3 Make a similar semistandard numbering for the frame $F^{321/21}$ and show that the same number of semistandard tableaux arise in the set of frames $F^3 + 2F^{21} + F^{1^3}$.

■ 3.2 Young tableaux and monomials

A numbered frame may be associated with a unique monomial by replacing each integer i by a variable x_i. Thus the Young tableau

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 2 & 4 & 5 \\
\hline
3 & 3 & 3 & 5 \\
\cline{1-4}
4 & 6 & 7 \\
\cline{1-3}
5 & 7 & 8 \\
\cline{1-3}
6 & 8 \\
\cline{1-2}
7 \\
\cline{1-1}
\end{array}$$

can be associated with the monomial $x_1^2\, x_2\, x_3^3\, x_4^2\, x_5^3\, x_6^2\, x_7^3\, x_8^2$

■ 3.3 Monomial symmetric functions

Consider a set of variables $(x) = x_1, x_2, \ldots, x_d$. A *symmetric* monomial

$$\boxed{\boxed{m_\lambda(x) = \sum_\alpha x^\alpha}} \tag{3.4}$$

involves a sum over all distinct permutations α of $(\lambda) = (\lambda_1, \lambda_2, \ldots)$. Thus if $(x) = (x_1, x_2, x_3)$ then

$$m_{21}(x) = x_1^2\, x_2 + x_1^2\, x_3 + x_1\, x_2^2 + x_1\, x_3^2 + x_2^2\, x_3$$
$$m_{1^3}(x) = x_1\, x_2\, x_3$$

The semistandard numbering of $(\lambda) = (2\,1)$ with 1, 2, 3 corresponds to the sum of monomials

$$\boxed{\boxed{m_{21}(x) + 2m_{1^3}(x)}} \tag{3.5}$$

The same linear combination occurs for any number of variables with $d \geq 3$.

The monomials $m_\lambda(x)$ are *symmetric functions.* If $\lambda \vdash n$ then $m_\lambda(x)$ is homogeneous of degree n. Unless otherwise stated we shall henceforth assume that x involves an infinite number of variables x_i.

The *ring of symmetric functions* $\Lambda = \Lambda(x)$ is the vector space spanned by all the $m_\lambda(x)$. This space can be decomposed as

$$\Lambda = \oplus_{n \geq 0} \Lambda^n \tag{3.6}$$

where Λ^n is the space spanned by all m_λ of degree n. Thus the $\{m_\lambda | \lambda \vdash n\}$ form a basis for the space Λ^n which is of dimension $p(n)$ where $p(n)$ is the number of partitions of n. It is of interest to ask if other bases can be constructed for the space Λ^n.

■ 3.4 The classical symmetric functions

Three other classical bases are well-known - some since the time of Newton.

1. The elementary symmetric functions
   The $n-$th elementary symmetric function e_n is the sum over all products of n distinct variables x_i, with $e_0 = 1$ and generally

$$e_n = m_{1^n} = \sum_{i_1 < i_2 \ldots < i_n} x_{i_1}\, x_{i_2}\, \ldots\, x_{i_n} \tag{3.7}$$

The *generating function* for the e_n is

$$E(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1}(1+x_i t) \tag{3.8}$$

2. The complete symmetric functions
   The $n-$th complete or *homogeneous* symmetric function h_n is the sum of all monomials of total degree n in the variables $x_1, x_2, \ldots,$ with $h_0 = 1$ and generally

$$h_n = \sum_{|\lambda|=n} m_\lambda = \sum_{i_1\leq i_2\ldots\leq i_n} x_{i_1}\, x_{i_2}\, \ldots\, x_{i_n} \tag{3.9}$$

The generating function for the h_n is

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1}(1-x_i t)^{-1} \tag{3.10}$$

3. The power sum symmetric function
   The $n-$th power sum symmetric function is

$$p_n = m_n = \sum_{i\geq 1} x_i^n \tag{3.11}$$

The generating function for the p_n is

$$\begin{aligned} P(t) = \sum_{n\geq 1} p_n t^{n-1} &= \sum_{i\geq 1}\sum_{n\geq 1} x_i^n t^{n-1} \\ &= \sum_{i\geq 1} \frac{x_i}{1-x_i t} \\ &= \sum_{i\geq 1} \frac{d}{dt}\log\frac{1}{1-x_i t} \end{aligned} \tag{3.12}$$

and hence

$$\begin{aligned} P(t) &= \frac{d}{dt}\log\prod_{i\geq 1}(1-x_i t)^{-1} \\ &= \frac{d}{dt}\log H(t) \\ &= H^{'}(t)/H(t) \end{aligned} \tag{3.13}$$

Similarly,

$$P(-t) = \frac{d}{dt}\log E(t) = E^{'}(t)/E(t) \tag{3.14}$$

Equation (3.13) leads to the relationship

$$nh_n = \sum_{r=1}^{n} p_r\, h_{n-r} \tag{3.15}$$

It follows from (3.13) that

$$\begin{aligned} H(t) &= \exp\sum_{n\geq 1} p_n\, t^n/n \\ &= \prod_{n\geq 1}\exp(p_n\, t^n/n) \\ &= \prod_{n\geq 1}\sum_{m_n=0}^{\infty}(p_n\, t^n)^{m_n}/n^{m_n}.m_n! \end{aligned} \tag{3.15}$$

and hence

$$H(t) = \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}\, t^{|\lambda|} \tag{3.16}$$

where

$$z_{\lambda} = \prod_{i \geq 1} i^{m_i}.m_i! \tag{3.17}$$

where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i.
Defining

$$\varepsilon_{\lambda} = (-1)^{|\lambda| - \ell(\lambda)} \tag{3.18}$$

we can show in an exactly similar manner to that of Eq.(3.16) that

$$E(t) = \sum_{\lambda} \varepsilon_{\lambda} z_{\lambda}^{-1} p_{\lambda}\, t^{|\lambda|} \tag{3.19}$$

It then follows from Eqs.(3.16) and (3.19) that

$$h_n = \sum_{|\lambda|=n} z_{\lambda}^{-1} p_{\lambda} \tag{3.20}$$

and

$$e_n = \sum_{|\lambda|=n} \varepsilon_{\lambda} z_{\lambda}^{-1} p_{\lambda} \tag{3.21}$$

■ Exercises

3.4 Show that for $n = 3$

$$\begin{aligned} p_3 &= x_1^3 + x_2^3 + x_3^3 + \dots \\ e_3 &= x_1\, x_2\, x_3 + x_1\, x_2\, x_4 + x_2\, x_3\, x_4 + \dots \\ h_3 &= x_1^3 + x_2^3 + \dots + x_1^2\, x_2 + x_1\, x_2^2 + \dots + x_1\, x_2\, x_3 + x_1\, x_2\, x_4 + \dots \end{aligned} \tag{3.22}$$

3.5 Noting Eqs. (3.8) and (3.10) and that $H(t)E(-t) = 1$, show that

$$\sum_{r=0}^{n} (-1)^r h_{n-r}\, e_r = 0 \tag{3.23}$$

for $n \geq 1$.

3.6 Use Eq.(3.15) to show that

$$e_n = det(h_{1-i+j})_{1 \leq i,j \leq n} \tag{3.24}$$

and hence

$$h_n = det(e_{1-i+j})_{1 \leq i,j \leq n} \tag{3.25}$$

3.7 Use Eq.(3.15) to obtain the determinantal expressions

$$p_n = \begin{vmatrix} e_1 & 1 & 0 & \dots & 0 \\ 2e_2 & e_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ ne_n & e_{n-1} & e_{n-2} & \dots & e_1 \end{vmatrix} \tag{3.26}$$

$$n!e_n = \begin{vmatrix} p_1 & 1 & 0 & \dots & 0 \\ p_2 & p_1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ p_{n-1} & p_{n-2} & . & \dots & n-1 \\ p_n & p_{n-1} & . & \dots & p_1 \end{vmatrix} \tag{3.27}$$

$$(-1)^{n-1}p_n = \begin{vmatrix} h_1 & 1 & 0 & \dots & 0 \\ 2h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ nh_n & h_{n-1} & h_{n-2} & \dots & h_1 \end{vmatrix} \tag{3.28}$$

$$n!h_n = \begin{vmatrix} p_1 & -1 & 0 & \dots & 0 \\ p_2 & p_1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ p_{n-1} & p_{n-2} & . & \dots & -n+1 \\ p_n & p_{n-1} & . & \dots & p_1 \end{vmatrix} \tag{3.29}$$

■ 3.5 Multiplicative bases for Λ^n

The three types of symmetric functions, h_n, e_n, p_n, do not have enough elements to form a basis for Λ^n, there must be one function for every partition $\lambda \vdash n$. To that end in each case we form *multiplicative* functions f_λ so that for each $\lambda \vdash n$

$$f_\lambda = f_{\lambda_1} f_{\lambda_2} \dots f_{\lambda_\ell} \tag{3.30}$$

where $f = e,\, h,$ or p Thus, for example,

$$e_{21} = e_2 \cdot e_1 = (x_1 x_2 + x_1 x_3 + x_2 x_3 + \dots)(x_1 + x_2 + x_3 + \dots)$$

■ 3.6 The Schur functions

The symmetric functions

$$m_\lambda,\, e_\lambda,\, h_\lambda,\, p_\lambda \tag{3.31}$$

where $\lambda \vdash n$ each form a basis for Λ^n. A very important fifth basis is realised in terms of the Schur functions, s_λ, *or for brevity, $S-$functions* which may be variously defined. Combinatorially they may be defined as

$$s_\lambda(x) = \sum_T x^T \tag{3.32}$$

where the summation is over all semistandard
$\lambda-$tableaux T. For example, consider the $S-$functions s_λ in just three variables $(x_1,\, x_2,\, x_3)$. For $\lambda = (2\,1)$ we have the eight tableaux T found earlier

$$\begin{array}{cccccccc} \begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix} & \begin{smallmatrix} 1 & 1 \\ 3 & \end{smallmatrix} & \begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix} & \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} & \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} & \begin{smallmatrix} 1 & 3 \\ 3 & \end{smallmatrix} & \begin{smallmatrix} 2 & 2 \\ 3 & \end{smallmatrix} & \begin{smallmatrix} 2 & 3 \\ 3 & \end{smallmatrix} \end{array} \tag{3.33}$$

Each tableaux T corresponds to a monomial x^T to give

$$\begin{aligned} s_{2\,1}(x_1,\, x_2,\, x_3) =& x_1^2\, x_2 + x_1^2\, x^3 + x_1\, x_2^2 + x_1\, x_2\, x_3 + x_1\, x_2\, x_3 + x_1\, x_3^2 \\ &+ x_2^2\, x_3 + x_2\, x_3^2 \end{aligned} \tag{3.34}$$

We note that the monomials in Eq.(3.34) can be expressed in terms of just two *symmetric monomials* in the three variables $(x_1,\, x_2,\, x_3)$ to give

$$s_{2\,1}(x_1,\, x_2,\, x_3) = m_{2\,1}(x_1,\, x_2,\, x_3) + 2m_{1^3}(x_1,\, x_2,\, x_3) \tag{3.35}$$

In an arbitrary number of variables

$$s_{2\,1}(x) = m_{2\,1}(x) + 2m_{1^3}(x) \tag{3.36}$$

This is an example of the general result that the
$S-$function may be expressed as a linear combination of symmetric monomials as indeed would be expected if the $S-$functions are a basis of Λ^n. In fact

$$s_\lambda(x) = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu \tag{3.37}$$

where $|\lambda| = n$ and $K_{\lambda\lambda} = 1$. The $K_{\lambda\mu}$ are the elements of an upper triangular matrix K known as the Kostka matrix. K is an example of a *transition matrix* that relates one symmetric function basis to another.

■ 3.7 Calculation of the elements of the Kostka matrix

The elements $K_{\lambda\mu}$ of the Kostka matrix may be readily calculated by the following algorithm :

i. Draw the frame F^λ.
ii. Form all possible semistandard tableaux that arise in numbering F^λ with μ_1 ones, μ_2 twos etc.
iii. $K_{\lambda\mu}$ is the number of semistandard tableaux so formed.

Thus calculating $K_{(42)\,(2^2\,1^2)}$ we obtain the four semistandard tableaux

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3&4\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&4\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\\hline 2&3\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline 2&2\\\cline{1-2}\end{array}$$

and hence $K_{(42)\,(2^2\,1^2)} = 4$.

■ Exercises

3.8 Construct the Kostka matrix for $\lambda, \mu \vdash 4$.

3.9 Show that in the variables $(x_1,\, x_2,\, x_3)$ the evaluation of the determinantal ratio

$$\frac{\begin{vmatrix} x_1^4 & x_1^2 & 1 \\ x_2^4 & x_2^2 & 1 \\ x_3^4 & x_3^2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix}}$$

yields the monomial content of the $S-$function s_{21} in three variables as found in Eq.(3.36). N.B. The above exercise is tedious by hand but trivial using MAPLE.

The last exercise is an example of the classical definition, as opposed to the equivalent combinatorial definition given in Eq.(3.32), given first by Jacobi, namely,

$$s_\lambda = s_\lambda(x_1,\, x_2,\, \ldots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta} \tag{3.38}$$

where λ is a partition of length $\leq n$ and
$\delta = (n-1,\, n-2,\, \ldots,\, 1,\, 0)$ with

$$a_{\lambda+\delta} = det(x_i^{\lambda_j+n-j})_{1\leq i,j\leq n} \tag{3.39}$$

and

$$a_\delta = \prod_{1\leq i,j\leq n} (x_i - x_j) = det(x_i^{n-j}) \tag{3.40}$$

is the *Vandermonde determinant.* Note that the Vandermonde determinant is an *alternating* or *antisymmetric* function. Any *even* power of the Vandermonde determinant is an *symmetric* function. This has important applications in the interpretation of the quantum Hall effect.

■ 3.8 Non-standard $S-$functions

The $S-$functions are symmetric functions indexed by ordered partitions λ. We shall frequently write $S-$functions $s_\lambda(x)$ as $\{\lambda\}(x)$ or, since we will generally consider the number of variables to be unrestricted, just $\{\lambda\}$. As a matter of notation the partitions will normally be written without spacing or commas separating the parts where $\lambda_i \leq 9$. A space will be left after any part $\lambda_i \geq 10$. Thus we write $\{12, 11, 9, 8, 3, 2, 1\} \equiv \{12\;\;11\;\;98321\}$ While we have defined the $S-$function in terms of ordered partitions we sometimes encounter $S-$functions that are not in the standard form and must convert such *non-standard* $S-$functions into standard $S-$functions. Inspection of the determinantal forms of the $S-$function leads to the establishment of the following *modification rules* :

$$\{\lambda_1, \lambda_2, \ldots, -\lambda_\ell\} = 0 \tag{3.41}$$

$$\{\lambda_1,\ldots,\lambda_i,\lambda_{i+1},\ldots,\lambda_\ell\} = -\{\lambda_1,\ldots,\lambda_{i+1}-1,\lambda_i+1,\ldots,\lambda_\ell\} \tag{3.42}$$

$$\{\lambda\} = 0 \qquad \text{if } \lambda_{i+1} = \lambda_i + 1 \tag{3.43}$$

Repeated application of the above three rules will reduce any non-standard $S-$function to either zero or to a signed standard $S-$function. In the process of using the above rules trailing zero parts are omitted

■ Exercise

3.10 Show that

$$\{24\} = -\{3^2\}, \quad \{141\} = -\{321\}$$
$$\{14-25-14\} = -\{3^3 2\}$$
$$\{3042\} = 0, \quad \{3043\} = \{3^2 2\}$$

■ 3.9 Skew $S-$functions

The combinatorial definition given for $S-$functions in Eq.(3.32) is equally valid for skew tableaux and can hence be used to define *skew* $S-$functions $s_{\lambda/\mu}(x)$ or $\{\lambda/\mu\}$. Since the $s_{\lambda/\mu}(x)$ are symmetric functions they must be expressible in terms of $S-$functions $s_\nu(x)$ such that

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu \tag{3.44}$$

It may be shown that the coefficients $c^\lambda_{\mu\nu}$ are necessarily non-negative integers and symmetric with respect to μ and ν. The coefficients $c^\lambda_{\mu\nu}$ are commonly referred to as *Littlewood-Richardson* coefficients.

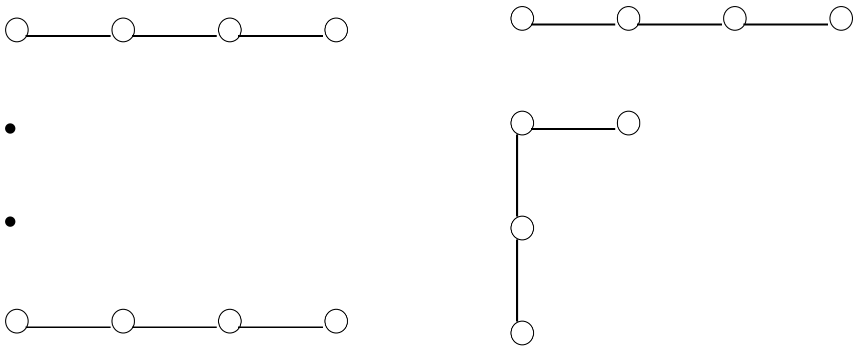
■ 3.10 Slinkies and Modification Rules

In situations involving extensive use of modification rules and in particular when one is trying to derive general formulae the use of *slinkies* can be very useful (KWY:King, Wybourne and Yang, *J. Phys. A: Math. Gen.* **22**, 4519 (1989)). (see also Chen, Garsia and Remmel, *Contemp. Math.* **34**, 109 (1984)). A slinky of length q is a diagram of q circles joined by $q-1$ links. A slinky can be folded so as to take the shape of a continuous boundary strip of a regular Young diagram, with each of the links eithehorizontal or vertical and its circles forming part of the boundary of such a diagram. The *sign* of the slinky is defined to be $(-1)^{r-1}$ where r is the number of rows occupied by the circles of the slinky, so that $r-1$ is the number of vertical links of the slinky.

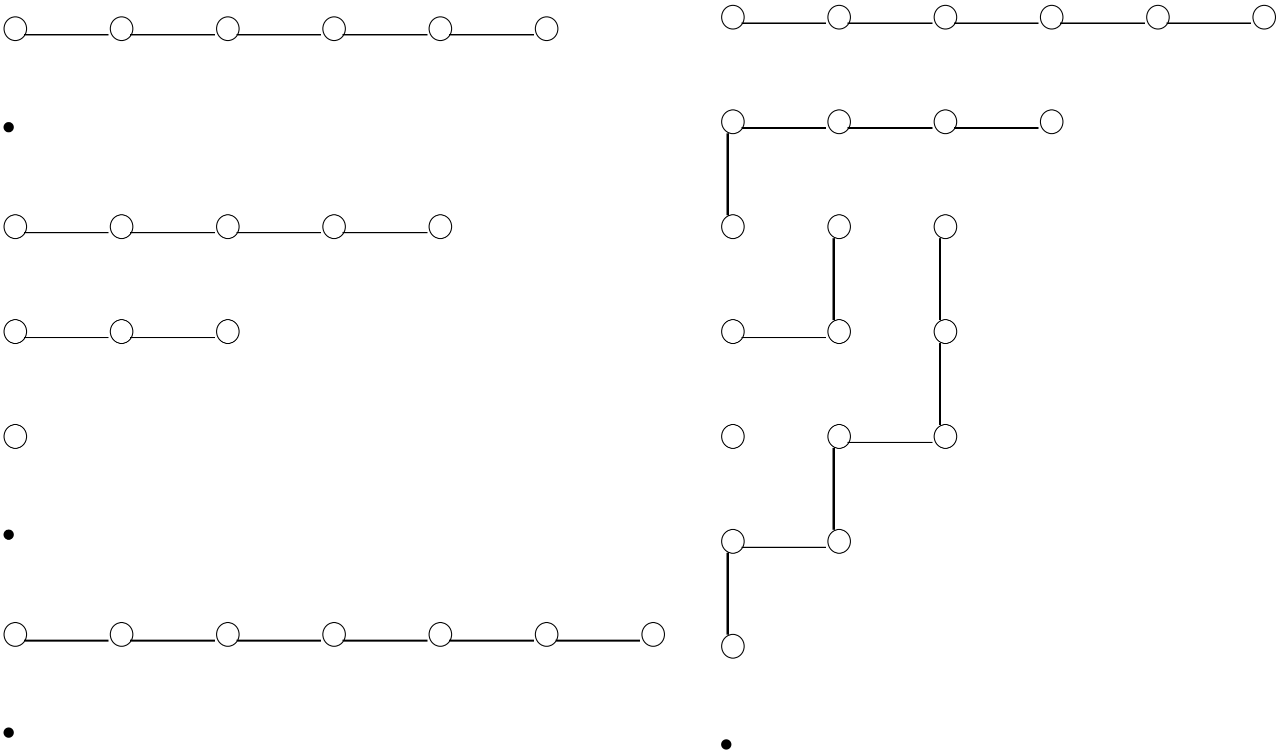
The modification rules for non-standard $S-$functions can be implemented in terms of folding operations of the slinkies that make up the Young diagram as follows:

1. Draw the slinky diagram corresponding to the non-standard $S-$function $\{\lambda_1,\lambda_2,\ldots,\lambda_p\}$.
2. Successively, for $i = 1,2,\ldots,p$, while holding the starting positions of the slinkies fixed, fold (if necessary) the $i-$th slinky of length λ_i into the shape of the unique standard continuous boundary strip such that the first i rows of the resulting diagram constitute a regular Young diagram. If this is not possible then $\{\lambda\} = 0$. Otherwise we obtain, after folding the last slinky, the regular Young diagram corresponding to some standard $S-$function $\{\mu\}$. The final result is then $\{\lambda\} = (-1)^v\{\mu\}$ where v is the total number of vertical links in the diagram.

We illustrate the application of the method of slinkies with two examples.



$\{4004\} \qquad \Rightarrow \qquad \{4211\}$



$\{60531070\} \qquad \Rightarrow \qquad \{64333210\}$

The principal application of the slinky method is to the expansion of symmetric generating functions as a sum of $S-$functions. Thus, for example, one (KWY) can show that

$$\prod_i (1 + x_i - xi^2) = \sum_{q,r=0}^{\infty} (-1)^q f_{r+1} \{2^q 1^r\}$$

where f_{r+1} is the $(r+1)-$th Fibonacci number.

■ Exercises

3.11 Show that

$$\{24\} = -\{3^2\}, \quad \{141\} = -\{321\}, \quad \{3042\} = 0, \quad \{3043\} = +\{3^2 2\}, \quad \{14-25-14\} = -\{3^3 2\}$$

3.12 Extend the slinky algorithm to include the possibility of negative parts and then show that $\{14-25-14\} = -\{3^3 2\}$.

3.13 Use the method of slinkies to show that

$$\{60531070\} = \{643^3 21\} \quad \text{and} \quad \{61131090\} = 0$$

■ General Remarks concerning $S-$functions

The $S-$functions are symmetric functions and form an integral basis for the ring of symmetric functions and hence may be expressed in terms of the classical symmetric functions $e_\lambda,\ h_\lambda,\ m_\lambda,\ f_\lambda$. Transition matrices can be defined for taking one from members of one basis to another. The transition matrices can be expressed in terms of the Kostka matrix $K_{\lambda\mu}$ and the transposition matrix

$$J_{\lambda\mu} = \begin{cases} 1, & \text{if} \tilde{\lambda} = \mu; \\ 0. & \text{otherwise} \end{cases} \tag{59}$$

The relevant transition matrices are tabulated in Macdonald (p56). These matrices all involve integers only.

The elementary and homogeneous symmetric functions e_n and h_n are special cases of $S-$functions , namely,

$$e_n \equiv \{1^n\} \qquad h_n \equiv \{n\} \tag{3.45}$$

$S-$functions arise in many situations. In the next few lectures we shall explore some of their properties that are relevant to applications in physics an chemistry. To proceed to these we must first consider the Littlewood-Richardson rule and then discuss the role of $S-$functions in the character theory of the symmetric group $S(n)$ and the unitary group $U(n)$.