Symmetric Functions and the Symmetric Group 2

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With the odd number five strange natures laws
Plays many freaks nor once mistakes the cause
And in the cowslap peeps this very day
Five spots appear which time neer wears away
Nor once mistakes the counting - look within
Each peep and five nor more nor less is seen
And trailing bindweed with its pinky cup
Five lines of paler hue goes streaking up
And birds a many keep the rule alive
And lay five eggs nor more nor less then five
And flowers how many own that mystic power
With five leaves making up the flower
John Clare ˜ 1821

2.1 Permutations and the Symmetric Group

Permutations play an important role in the physics of identical particles. A permutation leads to a reordering of a sequence of objects. We can place n objects in the natural number ordering $1, 2, \ldots, n$. Any other ordering can be discussed in terms of this ordering and can be specified in a two line notation

$$\begin{matrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{matrix} \tag{2.1}$$

For $n = 3$ we have the six permutations

$$\begin{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \end{matrix} \tag{2.2}$$

Permutations can be multiplied working from right to left. Thus

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

The six permutations in (2.2) satisfy the following properties:

1. There is an identity element $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.
2. Every element has an inverse among the set of elements.
2. The product of any two elements yields elements of the set.
4. The elements satisfy the associativity condition $a(bc) = (ab)c$. These conditions establish that the permutations form a *group*. In general the $n!$ permutations form the elements of the *symmetric group* $\mathcal{S}_n$.

■ Exercise 2.1 Construct a multiplication table (The Cayley Table) for the six permutations given in (2.2) and verify that the set of six permutations form a group.

■ Exercise 2.2 Inspect your Cayley table and see what subsets of the elements satisfy the four group axioms and thus form a *subgroup* of $\mathcal{S}_6$.

■ 2.2 Cycle Structure of Permutations

It is useful to express permutations as a cycle structure. A cycle $(i, j, k, \ldots, l)$ is interpreted as $i \to j, j \to k$ and finally $l \to i$. Thus our six permutations have the cycle structures

$$(1)(2)(3), (1,2)(3), (1)(2,3), (1,3)(2), (1,3,2), (1,2,3) \tag{2.3}$$

The elements within a cycle can be cyclically permuted and the order of the cycles is irrelevant. Thus $(123)(45) \equiv (54)(312)$.

■ A *k-cycle* or *cycle of length k* contains k elements. It is useful to organise cycles into *types* or *classes*. We shall designate the *cycle type* of a permutation π by

$$(1^{m_1} 2^{m_2} \ldots, n^{m_n}) \tag{2.4}$$

where m_k is the number of cycles of length k in the cycle representation of the permutation π.

■ For $\mathcal{S}_4$ there are five cycle types

$$(1^4), (1^2\ 2^1), (2^2), (1^1\ 3^1), (4^1) \tag{2.5}$$

Normally exponents of unity are omitted and Eq.(2.5) written as

$$(1^4), (1^2 2), (2^2), (13), (4) \tag{2.6}$$

■ Cycle types may be equally well labelled by ordered partitions of the integer n

$$\lambda = (\lambda_1 \lambda_2 \ldots \lambda_\ell) \tag{2.6}$$

where the λ_i are weakly decreasing and

$$\sum_{i=1}^{\ell} \lambda_i = n \tag{2.7}$$

The partition is said to be of *length* ℓ_λ and of *weight* $w_\lambda = n$. In terms of partitions the cycle types for $\mathcal{S}_5$ are

$$(1^5), (21^3), (2^2 1), (32), (31^2), (41), (5) \tag{2.8}$$

■ 2.3 Conjugacy Classes of $\mathcal{S}_n$

In any group G the elements g and h are *conjugates* if

$$g = khk^{-1} \qquad \text{for some} \qquad k \in G \tag{2.9}$$

The set of all elements conjugate to a given g is called the *conjugacy class* of g which we denote as K_g.

■ Exercises

2.3 Show that for $\mathcal{S}_4$ there are five conjugacy classes that may be labelled by the five partitions of the integer 4.

2.4 Show that the permutations, expressed in cycles, with cycles of length one suppressed, divide among the conjugacy classes as

$$\begin{aligned}
(1^4) &\supset e \\
(21^2) &\supset (12), (13), (14), (23), (24), (34) \\
(2^2) &\supset (12)(34), (13)(24), (14)(23) \\
(31) &\supset (123), (124), (132), (134), (142) \\
&\quad (143), (234), (243) \\
(4) &\supset (1234), (1243), (1342), (1432)
\end{aligned} \tag{2.10}$$

In general two permutations are in the same conjugacy class if, and only if, they are of the same cycle type. The number of classes of $\mathcal{S}_n$ is equal the number of partitions of the integer n.

If $\lambda = (1^{m_1}2^{m_2}\dots n^{m_n})$ then the number of permutations k_λ in the class (λ) of $\mathcal{S}_n$ is

$$k_\lambda = \frac{n!}{1^{m_1}m_1!2^{m_2}m_2!\dots n^{m_n}m_n!} \tag{2.11}$$

■ 2.4 The Cayley Table for $\mathcal{S}_3$

	e	(12)	(13)	(23)	(132)	(123)
e	e	(12)	(13)	(23)	(132)	(123)
(12)	(12)	e	(132)	(123)	(13)	(23)
(13)	(13)	(123)	e	(132)	(23)	(12)
(23)	(23)	(132)	(123)	e	(12)	(13)
(132)	(132)	(23)	(12)	(13)	(123)	e
(123)	(123)	(13)	(23)	(12)	e	(132)

■ 2.5 Transpositions and cycles of $\mathcal{S}_n$

1. A cycle of order two is termed a *transposition.*
2. A transposition $(i,\, i+1)$ is termed an *adjacent transposition.*
3. The entire symmetric group $\mathcal{S}_n$ can be generated (or given a *presentation* in terms of the set of adjacent transpositions

$$(1\,2),\ (2\,3),\ \dots,\ (n-1\,n) \tag{2.12}$$

■ If $\pi = \tau_1\tau_2\dots\tau_k$, where the τ_i are transpositions then the *sign* of π is defined to be

$$sgn(\pi) = (-1)^k \tag{2.13}$$

If the number of cycles of *even* order is *even* then the permutation is *even* or *positive*; if it is *odd* then the permutation is *odd* or *negative.*

■ 2.6 The Presentation of $\mathcal{S}_n$

Let us designate an adjacent transposition by

$$s_i = (i, i+1) \qquad \text{for} \quad i = 1, 2, \dots, n-1 \tag{2.14}$$

then we can give a *presentation* of the symmetric group $\mathcal{S}_n$ in terms of the s_i via the following three relations:-

$$s_i^2 = 1 \qquad \text{for} \quad i = 1, 2, \dots, n-1 \tag{2.15a}$$

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \qquad \text{for} \quad i = 1, 2, \dots, n-2 \tag{2.15b}$$

$$s_is_j = s_js_i \qquad \text{for} \quad |i-j| \geq 2 \tag{2.15c}$$

Every permutation π in S_n can be expressed as a *reduced word* of minimal length $\ell(\pi)$ in the s_i.

■ Exercise

2.5 Verify the last sentence in the case of $\mathcal{S}_3$

■ 2.7 Note on Hecke algebra $\mathcal{H}_n(q)$ of type $\mathcal{A}_{n-1}$

We can $q-$deform the presentation of $\mathcal{S}_n$ to give the complex Hecke algebra $\mathcal{H}_n(q)$, with q an arbitrary but fixed complex parameter, generated by g_i with $i = 1, 2, \dots, n-1$ subject to the relations:

$$g_i^2 = (q-1)g_i + q \qquad \text{for} \quad i = 1, 2, \dots, n-1 \tag{2.16a}$$

$$g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1} \qquad \text{for} \quad i = 1, 2, \dots, n-2 \tag{2.16b}$$

$$g_ig_j = g_jg_i \qquad \text{for} \quad |i-j| \geq 2 \tag{2.16c}$$

For $q = 1$ these relations are exactly those appropriate to the symmetric group $\mathcal{S}_n$. There exists a map h from $\mathcal{S}_n$ to $\mathcal{H}_n(q)$ such that $h(s_i) = g_i$ and $h(\pi) = g_{i_1}g_{i_2}\dots g_{i_m}$ for any permutation $\pi = s_{i_1}s_{i_2}\dots s_{i_m} \in$

$\mathcal{S}_n$. The set of reduced words $h(\pi)$ for all $n!$ permutations $\pi \in \mathcal{S}_n$ forms a basis of $\mathcal{H}_n(q)$. For more details see:- R. C. King and B. G. Wybourne, *J. Phys. A: Math. Gen. 23* L1193 (1990).

■ 2.8 The Alternating Group $\mathcal{A}_n$

The set of *even* permutations form a subgroup of $\mathcal{S}_n$ known as the *alternating group* $\mathcal{A}_n$ and has precisely half the elements of $\mathcal{S}_n$ i.e. $(\frac{1}{2})n!$.

■ Exercises

2.6 Show that the set of six matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \tag{3.17}$$

with the usual rule of matrix multiplication form a group isomorphic to $\mathcal{S}_3$.

2.7 Show that the symmetric group $\mathcal{S}_n$ has two one-dimensional representations, a symmetric representation where every element is mapped onto
unity and an antisymmetric representation where the elements are mapped onto the sign defined in Eq. (2.13).