Symmetric Functions and the Symmetric Group 11
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It was a dark and stormy night when R. H. Bing volunteered to drive some stranded mathematicians from the fogged-in Madison airport to Chicago. Freezing rain pelted the windscreen and iced the roadway as Bing drove on - concentrating deeply on the mathematical theorem he was explaining. Soon the windshield was fogged from the energetic explanation. The passengers too had beaded brows, but their sweat rose from fear. As the mathematical description got brighter, the visibility got dimmer. Finally, the conferees felt a trace of hope for their survival when Bing reached forward - apparently to wipe off the moisture from the windshield. Their hope turned to horror when, instead, Bing drew a figure with his finger on the foggy pane and continued his proof - embellishing the illustration with arrows and helpful labels as needed for the demonstration.

— Prof. R H Bing, famous US topologist

■ 10.1 Introduction

Today I want to remark on properties of bosons and fermions. Recall that bosons are objects with integer spin and that the wavefunction for $N-$identical bosons is totally symmetric while fermions are objects with half-integer spin and that the wavefunction for $N-$identical fermions is totally antisymmetric. Note the use of the word *totally.* Here we will be considering identical fermions, or bosons, in an isotropic harmonic oscillator. The $N-$particle wavefunction is the product of a spin part with a spatial part and symmetrization involves both parts. Thus a $N-$particle wavefunction maybe totally antisymmetric (*fermions*) or totally symmetric (*bosons*) yet the spin or spatial parts need not be, their product *must* be. I will start by going through the analysis of the isotropic harmonic oscillator for a single particle and then sketch the problem of enumerating states for the case of $N-$noninteracting particles - bosons or fermions. Much of what we have to say in this lecture is applicable to nuclei, quantum dots and statistical physics.

■ 10.2 The isotropic three-dimensional harmonic oscillator

For a one-dimensional harmonic oscillator one has the well-known energy spectrum (putting $\hbar = 1$)

$$E_n = (n + \tfrac{1}{2})\nu \qquad n = 0, 1, \ldots \tag{1}$$

where ν is the usual frequency. We can regard an isotropic three-dimensional harmonic oscillator as three one-dimensional harmonic oscillator each of frequency ν and energy eigenvalues

$$E_n = (n + \tfrac{3}{2})\nu \qquad n = 0, 1, \ldots \tag{2}$$

with

$$n = n_x + n_y + n_z \qquad n_x, n_y, n_z = 0, 1, \ldots \tag{3}$$

Each state may be labelled by the triplet of numbers (n_x, n_y, n_z) and for a given value of n we will obtain $(n+1)(n+2)/2$ distinct triplets (n_x, n_y, n_z). For example, the first four levels are associated with the states given below and corresponding to degeneracies, 1, 3, 6, 10. The triplets of quantum numbers can be regarded as the *weights* of the irreducible representations $\{n\}$ of the degeneracy group $U(3)$. As the isotropic three-dimensional harmonic oscillator is clearly rotationally invariant we could extend our description of our states by using the group chain

$$U(3) \supset SO(3) \supset SO(2) \tag{4}$$

and uniquely label the states by the set of quantum numbers $|n\ell m_\ell\rangle$, noting that for n odd and even we have the $U(3) \to SO(3)$ decompositions

$$n \to [n] + [n-2] + \ldots + \begin{cases} [0] & \text{if } n \text{ is even} \\ [1] & \text{if } n \text{ is odd} \end{cases} \tag{5}$$

Table 1. The values of the triplets (n_x, n_y, n_z) for the first four levels of an isotropic three-dimensional harmonic oscillator.

n	(n_x, n_y, n_z)
0	$(0,0,0)$
1	$(1,0,0)$
	$(0,1,0)$
	$(0,0,1)$
2	$(2,0,0)$
	$(0,2,0)$
	$(0,0,2)$
	$(1,1,0)$
	$(1,0,1)$
	$(0,1,1)$
3	$(3,0,0)$
	$(0,3,0)$
	$(0,0,3)$
	$(2,1,0)$
	$(2,0,1)$
	$(0,2,1)$
	$(0,1,2)$
	$(1,2,0)$
	$(1,0,2)$
	$(1,1,1)$

So far we have ignored the spin of our single particle. It is the spin of our particle that determines whether we are considering fermions or bosons. A complete description of the one-particle states requires the complete set of quantum numbers $|sm_s n\ell m_\ell\rangle$. This increases the degeneracies by a factor of $(2s+1)$ and extends the degeneracy group to $SU(2) \times U(3)$. Let us now look carefully at the case of two identical noninteracting fermions or bosons in a one-dimensional harmonic oscillator.

Two-particles in a one-dimensional harmonic oscillator

Let us suppose we have a boson with spin $s = 0$ and a fermion with spin $s = \frac{1}{2}$. Placed in a one-dimensional harmonic oscillator we have an infinite set of equi-spaced energy levels that may be indexed by integers $m = 0, 1, \ldots$. Each level has a spatial degeneracy of 1 and a spin degeneracy of $2s+1$. The degeneracy group is $SU(2) \times U(1)$ with each level labelled as ${}^{2s+1}\{m\}$ with m labelling the one-dimensional irreducible representations of $U(1)$. The complete set of one-particle states span the infinite set of $SU(2) \times U(1)$ irreducible representations that may be succinctly written as

$$\{2s\} \times M, \tag{6}$$

where

$$M = \sum_{m=0}^{\infty} \{m\}. \tag{7}$$

Now consider we place two noninteracting bosons (or fermions) in a one-dimensional harmonic oscillator. The bosonic two-particle states must be totally symmetric and the fermionic states totally antisymmetric with respect to the spin and spatial variables.

For the bosons the states all have $S = 0$ and the spatial symmetries come from extracting the symmetric part of $M \times M$ under $U(1)$ to give

$$\{M \times M\}_{sym} = \sum_{n=0}^{\infty} g^n \{n\}, \tag{8}$$

where

$$g^n = \left[\frac{n}{2}\right] + 1, \tag{9}$$

with $\left[\frac{n}{2}\right]$ is the integer part of $n/2$. Thus the degeneracy of the two-particle boson state labelled as n will be g^n.

Now consider the two-fermion states. These will have spin $S = 0$ (spin singlets) or $S = 1$ (spin triplets). At the $U(1)$ level the spatial part for the spin singlets will involve the symmetric part of $M \times M$ and hence give rise to the same states as in Eq.(8) while for the spin triplets the spatial part will involve the antisymmetric part of $M \times M$ with

$$\{M \times M\}_{anti} = \sum_{p=1}^{\infty} c^p \{p\}, \tag{10}$$

where

$$c^p = \left[\frac{p-1}{2}\right] + 1. \tag{11}$$

Clearly,

$$c^p = g^{p-1}. \tag{12}$$

Thus there is a one-to-one correspondence (relative to their respective ground states) between multiplicities of the $S = 0$ two-particle states of a pair of identical bosons or fermions. Likewise there is a one-to-one correspondence between the multiplicities of the $S = 0$ states of a pair of identical bosons and those of the $S = 1$ states of a pair of identical fermions shifted according to Eq.(12).

Results of (8) and (10) follow directly from

K Grudzinski and B G Wybourne, J. Phys. A:Math.Gen.**29**,6631 (1996).

N noninteracting particles in a one-dimensional harmonic oscillator

For $N-$noninteracting bosons in a one-dimensional harmonic oscillator one simply enumerates the $U(1)$ content of the symmetric part of the $N-$fold product $M^{\otimes\{N\}}$ to find that

$$M^{\otimes\{N\}} = \sum_{k=0}^{\infty} g_N^k \{k\} \tag{13}$$

where g_N^k is the number of partitions of k into at most N parts allowing repetitions and null parts. For example,

$$\begin{aligned} M^{\otimes\{4\}} \supset & \{0\} + \{1\} + 2\{2\} + 3\{3\} + 5\{4\} + 6\{5\} + 9\{6\} + 11\{7\} \\ & + 15\{8\} + 18\{9\} + 23\{10\} + \dots \end{aligned} \tag{14}$$

Now consider $N-$noninteracting fermions. To describe the multiplicities of the levels we need the antisymmetric $SU(2) \times U(1)$ content of

$$(\{1\} \times M)^{\otimes\{1^N\}} = \sum_{\sigma \vdash N} \{\sigma\} \times M^{\otimes\{\sigma'\}} \tag{15}$$

where the sum is over all partitions $(\sigma) = (\sigma_1, \sigma_2)$ of N into at most two parts with (σ') being the partition conjugate to (σ), involving partitions whose Young frame involves at most two columns. Thus for $N = 4$ we would have

$$(\{1\} \times M)^{\otimes\{1^4\}} = \{4\} \times M^{\otimes\{1^4\}} + \{31\} \times M^{\otimes\{21^2\}} + \{2^2\} \times M^{\otimes\{2^2\}} \tag{16}$$

The spin S_σ to be associated with a given partition (σ_1, σ_2) is

$$S_\sigma = \frac{\sigma_1 - \sigma_2}{2} \tag{17}$$

Under $U(1)$ the tensor products, say $\{p\} \times \{q\}$, have the simple form

$$\{p\} \times \{q\} = \{p + q\} \tag{18}$$

Using this fact one can show that under $U(1)$

$$M^{\otimes\{1^N\}} = \sum_{\ell=\frac{N(N-1)}{2}}^{\infty} c_N^\ell\{\ell\} \tag{19}$$

where c_N^ℓ is the number of partitions of the integer ℓ into N distinct parts including the null part. In fact

$$c_N^\ell = g_N^k \qquad \text{if} \quad \ell = k + \tfrac{N(N-1)}{2} \tag{20}$$

This can be seen by noting that we can map the sets of partitions into the other by adding, or subtracting, $\rho_N = (N-1,\ldots,2,1,0)$. Adding ρ_N to the partitions of k into at most N parts converts them into partitions, all of whose parts are distinct.

Thus in a one-dimensional harmonic oscillator there is a one-to-one correspondence between the multiplicities of the states of N identical bosons and those of the maximal spin states of N identical fermions. For four identical fermions we obtain the $U(1)$ content for the states with $S = 2$

$$\begin{aligned} M^{\otimes\{1^4\}} \supset &\{6\} + \{7\} + 2\{8\} + 3\{9\} + 5\{10\} + 6\{11\} + 9\{12\} \\ &+ 11\{13\} + 15\{14\} + 18\{15\} + 23\{16\} + \ldots \end{aligned} \tag{21}$$

which may be compared with Eq.(14).

However, the spatially antisymmetric states are not the complete set of states for N noninteracting fermions, one also has the mixed symmetry states associated with the other spin states. For example, for $N = 4$ we also have the $U(1)$ states with spin $S = 1$ coming from

$$\begin{aligned} M^{\otimes\{21^2\}} \supset &\{3\} + 2\{4\} + 4\{5\} + 6\{6\} + 10\{7\} + 14\{8\} + 20\{9\} \\ &+ 26\{10\} + 35\{11\} + 44\{12\} + 56\{13\} + 68\{14\} \\ &+ 84\{15\} + 100\{16\} + \ldots \end{aligned} \tag{22}$$

and for the spin $S = 0$ states

$$\begin{aligned} M^{\otimes\{2^2\}} \supset &\{2\} + \{3\} + 3\{4\} + 4\{5\} + 7\{6\} + 9\{7\} + 14\{8\} + 17\{9\} \\ &+ 24\{10\} + 29\{11\} + 38\{12\} + 45\{13\} + 57\{14\} \\ &+ 66\{15\} + 81\{16\} + \ldots \end{aligned} \tag{23}$$

N noninteracting particles in an isotropic three-dimensional harmonic oscillator

In this case the degeneracy group is $SU(2)\times U(3)$. Let us consider a single fermion of spin $s = \frac{1}{2}$. The spin spans the $\{1\}$ irreducible representation of $SU(2)$ while the orbital states span the irreducible representations $\{m\}$ of $U(3)$ with $m = 0, 1, \ldots, \infty$. Thus for a single fermion the complete set of states span the reducible representation $\{1\} \times M$ of $SU(2) \times U(3)$ with

$$M = \sum_{m=0}^{\infty} \{m\} \tag{24}$$

Let us set the groundstate energy to zero and assume the levels are equi-spaced by an energy ΔE. For N noninteracting particles in an isotropic three-dimensional harmonic oscillator we obtain the totally antisymmetric states, under $SU(2) \times U(3)$,

$$(\{1\} \times M) \otimes \{1^N\} = \sum_{m\geq n\geq 0, m+n=N}^{N} \{m-n\} \times (M \otimes \{2^n 1^{m-n}\}) \tag{25}$$

$$= \sum_{m\geq n\geq 0, m+n=N}^{N} {}^{(m-n+1)}(M \otimes \{2^n 1^{m-n}\}) \tag{26}$$

$$= \sum_{S_{min}}^{\frac{N}{2}} {}^{2S+1}(M \otimes \{2^{\frac{N}{2}+S} 1^{2S}\}) \tag{27}$$

where

$$S_{min} = \begin{cases} \frac{1}{2} & \text{if } N \text{ is odd} \\ 0 & \text{if } N \text{ is even} \end{cases} \tag{28}$$

The spin multiplicity, $(2S+1)$, is given as a left superscript in (26) and (27).

For four identical, noninteracting, fermions of spin $\frac{1}{2}$ we obtain, to weight 10 in $U(3)$ the states:-

$$\begin{array}{lllll}
{}^1\{2\} & +\ 2\ {}^1\{21\} & +\ {}^1\{21^2\} & +\ 3\ {}^1\{2^2\} & +\ 2\ {}^1\{2^21\} \\
+\ {}^1\{2^3\} & +\ {}^1\{3\} & +\ 3\ {}^1\{31\} & +\ 3\ {}^1\{31^2\} & +\ 5\ {}^1\{32\} \\
+\ 6\ {}^1\{321\} & +\ 3\ {}^1\{32^2\} & +\ 2\ {}^1\{3^2\} & +\ 5\ {}^1\{3^21\} & +\ 3\ {}^1\{3^22\} \\
+\ {}^1\{3^3\} & +\ 3\ {}^1\{4\} & +\ 7\ {}^1\{41\} & +\ 5\ {}^1\{41^2\} & +\ 12\ {}^1\{42\} \\
+\ 12\ {}^1\{421\} & +\ 8\ {}^1\{42^2\} & +\ 11\ {}^1\{43\} & +\ 14\ {}^1\{431\} & +\ 12\ {}^1\{432\} \\
+\ 5\ {}^1\{43^2\} & +\ 11\ {}^1\{4^2\} & +\ 11\ {}^1\{4^21\} & +\ 12\ {}^1\{4^22\} & +\ 4\ {}^1\{5\} \\
+\ 11\ {}^1\{51\} & +\ 10\ {}^1\{51^2\} & +\ 19\ {}^1\{52\} & +\ 22\ {}^1\{521\} & +\ 13\ {}^1\{52^2\} \\
+\ 21\ {}^1\{53\} & +\ 30\ {}^1\{531\} & +\ 24\ {}^1\{532\} & +\ 24\ {}^1\{54\} & +\ 32\ {}^1\{541\} \\
+\ 14\ {}^1\{5^2\} & +\ 7\ {}^1\{6\} & +\ 18\ {}^1\{61\} & +\ 15\ {}^1\{61^2\} & +\ 32\ {}^1\{62\} \\
+\ 36\ {}^1\{621\} & +\ 24\ {}^1\{62^2\} & +\ 39\ {}^1\{63\} & +\ 52\ {}^1\{631\} & +\ 48\ {}^1\{64\} \\
+\ 9\ {}^1\{7\} & +\ 25\ {}^1\{71\} & +\ 23\ {}^1\{71^2\} & +\ 45\ {}^1\{72\} & +\ 54\ {}^1\{721\} \\
+\ 59\ {}^1\{73\} & +\ 14\ {}^1\{8\} & +\ 37\ {}^1\{81\} & +\ 32\ {}^1\{81^2\} & +\ 67\ {}^1\{82\} \\
+\ 17\ {}^1\{9\} & +\ 48\ {}^1\{91\} & +\ 24\ {}^1\{10\ \} & +\ {}^3\{1^2\} & +\ {}^3\{1^3\} \\
+\ 2\ {}^3\{21\} & +\ 3\ {}^3\{21^2\} & +\ {}^3\{2^2\} & +\ 2\ {}^3\{2^21\} & +\ {}^3\{3\} \\
+\ 5\ {}^3\{31\} & +\ 6\ {}^3\{31^2\} & +\ 6\ {}^3\{32\} & +\ 8\ {}^3\{321\} & +\ 3\ {}^3\{32^2\} \\
+\ 7\ {}^3\{3^2\} & +\ 9\ {}^3\{3^21\} & +\ 7\ {}^3\{3^22\} & +\ 4\ {}^3\{3^3\} & +\ 2\ {}^3\{4\} \\
+\ 9\ {}^3\{41\} & +\ 11\ {}^3\{41^2\} & +\ 12\ {}^3\{42\} & +\ 17\ {}^3\{421\} & +\ 7\ {}^3\{42^2\} \\
+\ 16\ {}^3\{43\} & +\ 23\ {}^3\{431\} & +\ 17\ {}^3\{432\} & +\ 11\ {}^3\{43^2\} & +\ 10\ {}^3\{4^2\} \\
+\ 16\ {}^3\{4^21\} & +\ 12\ {}^3\{4^22\} & +\ 4\ {}^3\{5\} & +\ 16\ {}^3\{51\} & +\ 18\ {}^3\{51^2\} \\
+\ 24\ {}^3\{52\} & +\ 32\ {}^3\{521\} & +\ 16\ {}^3\{52^2\} & +\ 34\ {}^3\{53\} & +\ 48\ {}^3\{531\} \\
+\ 38\ {}^3\{532\} & +\ 32\ {}^3\{54\} & +\ 48\ {}^3\{541\} & +\ 26\ {}^3\{5^2\} & +\ 6\ {}^3\{6\} \\
+\ 24\ {}^3\{61\} & +\ 28\ {}^3\{61^2\} & +\ 38\ {}^3\{62\} & +\ 52\ {}^3\{621\} & +\ 27\ {}^3\{62^2\} \\
+\ 56\ {}^3\{63\} & +\ 82\ {}^3\{631\} & +\ 60\ {}^3\{64\} & +\ 10\ {}^3\{7\} & +\ 36\ {}^3\{71\} \\
+\ 40\ {}^3\{71^2\} & +\ 60\ {}^3\{72\} & +\ 80\ {}^3\{721\} & +\ 90\ {}^3\{73\} & +\ 14\ {}^3\{8\} \\
+\ 50\ {}^3\{81\} & +\ 56\ {}^3\{81^2\} & +\ 85\ {}^3\{82\} & +\ 20\ {}^3\{9\} & +\ 69\ {}^3\{91\} \\
+\ 26\ {}^3\{10\ \} & +\ {}^5\{1^3\} & +\ {}^5\{21^2\} & +\ {}^5\{31\} & +\ 3\ {}^5\{31^2\} \\
+\ {}^5\{32\} & +\ 2\ {}^5\{321\} & +\ 2\ {}^5\{3^2\} & +\ 4\ {}^5\{3^21\} & +\ 2\ {}^5\{3^22\} \\
+\ 3\ {}^5\{3^3\} & +\ 2\ {}^5\{41\} & +\ 5\ {}^5\{41^2\} & +\ 3\ {}^5\{42\} & +\ 5\ {}^5\{421\} \\
+\ {}^5\{42^2\} & +\ 5\ {}^5\{43\} & +\ 9\ {}^5\{431\} & +\ 5\ {}^5\{432\} & +\ 5\ {}^5\{43^2\} \\
+\ 3\ {}^5\{4^2\} & +\ 5\ {}^5\{4^21\} & +\ 3\ {}^5\{4^22\} & +\ 3\ {}^5\{51\} & +\ 8\ {}^5\{51^2\} \\
+\ 5\ {}^5\{52\} & +\ 10\ {}^5\{521\} & +\ 3\ {}^5\{52^2\} & +\ 9\ {}^5\{53\} & +\ 18\ {}^5\{531\} \\
+\ 12\ {}^5\{532\} & +\ 8\ {}^5\{54\} & +\ 16\ {}^5\{541\} & +\ 6\ {}^5\{5^2\} & +\ {}^5\{6\} \\
+\ 6\ {}^5\{61\} & +\ 11\ {}^5\{61^2\} & +\ 10\ {}^5\{62\} & +\ 16\ {}^5\{621\} & +\ 6\ {}^5\{62^2\} \\
+\ 17\ {}^5\{63\} & +\ 29\ {}^5\{631\} & +\ 18\ {}^5\{64\} & +\ {}^5\{7\} & +\ 9\ {}^5\{71\} \\
+\ 17\ {}^5\{71^2\} & +\ 15\ {}^5\{72\} & +\ 26\ {}^5\{721\} & +\ 26\ {}^5\{73\} & +\ 2\ {}^5\{8\} \\
+\ 13\ {}^5\{81\} & +\ 22\ {}^5\{81^2\} & +\ 23\ {}^5\{82\} & +\ 3\ {}^5\{9\} & +\ 18\ {}^5\{91\} \\
+\ 5\ {}^5\{10\ \} & +\ldots & & &
\end{array} \tag{29}$$

Terms having $U(3)$ irreducible representations, $\{\lambda\}$, of the same weight, $|\lambda|$, will have the same energy, E_λ, relative to the groundstate and independent of their spin multiplicity $(2S+1)$. A state $^{(}2S+1)\{\lambda\}$ will have

$$E_\lambda = |\lambda| \Delta E \tag{30}$$

For our four-fermion example the groundstate involves the 15 states arising from $^1\{2\} +^3 \{1^2\}$. The 6 states coming from $^1\{2\}$ can be viewed as arising from putting 2 fermions in the $1s$ orbitals and 2 in $1p$ orbitals to give rise to the two spectroscopic terms ^1SD while the 9 states coming from $^3\{1^2\}$ can likewise be viewed as arising from putting 2 fermions in the $1s$ orbitals and 2 in $1p$ orbitals but this time the orbital space is antisymmetric and the spin space symmetric and hence forming the spectroscopic term ^3P. NB. In atomic spectroscopy the one-electron states are given the traditional $n\ell$ labels viz.

$$1s, 2s, 3s, \ldots, 2p, 3p, 4p, \ldots, 3d, 4d, 5d, \ldots, 4f, 5f, 6f, \ldots \tag{31}$$

whereas in nuclear shell theory, where the isotropic three-dimensional harmonic oscillator is a useful starting point, the convention is to label the one-nucleon orbits with

$$1s, 2s, 3s, \ldots, 1p, 2p, 3p, \ldots, 1d, 2d, 3d, \ldots, 1f, 2f, 3f, \ldots \tag{32}$$

Here we follow the latter convention.

The next level involves the terms

$$^1(2\{21\} + \{3\}) +^3 (\{1^3\} + 2\{21\} + \{3\}) +^5 \{1^3\} \tag{31}$$

and hence a total degeneracy of 112. These are just the spectroscopic terms arising from the configurations $(1s)^2(1p)(2s)$, $(1s)^2(1p)(1d)$, $(1s)(1p)^3$.

In the above we have indicated how to count terms etc without calling upon the properties of the non-compact groups. That topic is largely covered in

K Grudziński and B G Wybourne, *Symplectic models of $n-$particle systems*, Rep. Math. Phys. **38**, 251-266 (1996).