# The Vandermonde Determinant Revisited

**Brian G Wybourne**

**Instytut Fizyki,**

**Uniwersytet Mikołaja Kopernika - Poland**

**11-18 September 2002**

**Dedicated to the memory of Claude Itzykson (1938-1995)**

## Introduction

- Claude Itzykson (1938-1995)
- The Laughlin Wavefunction
- Admissible Partitions
- The q-discriminant
- q-polynomials vanishing coefficients
- Some Results and Conjectures
- Hankel Determinants and Powers of the Vandermonde
- Concluding Remarks

cea
COMMISSARIAT A L'ÉNERGIE ATOMIQUE – DIRECTION DES SCIENCES DE LA MATIERE
SERVICE DE PHYSIQUE THÉORIQUE
CENTRE D'ÉTUDES DE SACLAY - ORME DES MERISIERS

Adresse Postale :
SERVICE DE PHYSIQUE THEORIQUE DE SACLAY
91191 GIF-SUR-YVETTE CEDEX, France
Téléphone : (1) 69 08 7269
Secrétariat : (1) 69 08 73 85
Fac-similé : (1) 69 08 81 20
Télex : ENERG 604641 F
Courrier électronique :

C. ITZYKSON

March 1 1994

Dear Pr. Wybourne

We have just received today your very interesting contribution to the expansion of (powers of) the discriminant. Indeed you seem to have discovered quite a striking phenomenon which deserves an explanation – Our preprint was thought too mathematical for Nucl. Phys. B. so it is resubmitted to Journ. Mod. Phys. A published in Singapore. I hope it will be found suitable there –

The subject seems to me to be still widely open = For instance is there a rule for the signs of the coefficients? What is the meaning of the vanishing terms you have found? What is their general feature? Is it significant that you found them starting at N = 8 ? etc...

Best wishes

Claude Itzykson

## Claude Itzykson's Comments

- Indeed you seem to have discovered a striking phenomenon which deserves an explanation ...
- The subject seems to me to be still widely open = For instance, is there a rule for the signs of the coefficients?
- What is the meaning of the vanishing terms you have found?
- What is their general feature?
- Is it significant that you found them starting at $N = 8$? etc..
- Claude Itzykson March 1 1994

## The Laughlin Wavefunction

- Laughlin[1] has described the fractional quantum Hall effect in terms of a wavefunction

$$\Psi^m_{Laughlin}(z_1,\ldots,z_N) = \prod_{i<j}^{N}(z_i - z_j)^{2m+1}\exp\left(-\frac{1}{2}\sum_{i=1}^{N}|z_i|^2\right) \quad (1)$$

- The Vandermonde alternating function in $N$ variables is defined

$$V(z_1,\ldots,z_N) = \prod_{i<j}^{N}(z_i - z_j) \quad (2)$$

$$\frac{\Psi^m_{Laughlin}}{V} = V^{2m} = \sum_{\lambda\vdash n} c^\lambda s_\lambda \quad n = mN(N-1) \quad (3)$$

where the $s_\lambda$ are Schur functions and the $c^\lambda$ are signed integers.

## Expansion of the Laughlin Wavefunction

- Dunne[2] and Di Francesco *et al*[3] have discussed properties of the expansions while Scharf *et al*[4] have given specific algorithms. I have extended their results to $N = 10$.
- The partitions $(\lambda)$ indexing the Schur functions are of weight $N(N-1)$. For a given $N$ the partitions are bounded by a highest partition $(2N-2, 2N-4, \ldots, 0)$ and a lowest partition $((N-1)^{N-1})$ with the partitions being of length $N$ and $N-1$.
- Let

$$n_k = \sum_{i=0}^{k} \lambda_{N-i} - k(k+1) \quad k = 0, 1, \ldots, N-1 \tag{4}$$

## Admissible Partitions

- Define[3] *Admissible partitions* as satisfying Eq(4) with *all* $n_k \geq 0$.
- Di Francesco *et al* conjectured that the number of admissible partitions, $A_N$, was the number of distinct partitions arising in the expansion, Eq(3), *provided none of the coefficients vanished.*
- The conjecture fails[4] for $N \geq 8$. We find the number of admissible partitions associated with vanishing coefficients as

$$(N = 8) \quad 8, \;(N = 9) \quad 66, \;(N = 10) \quad 389$$

- The coefficients of $s_\lambda$ and $s_{\lambda_r}$ are equal if[2]

$$(\lambda_r) = (2(N-1) - \lambda_N, \ldots, 2(N-1) - \lambda_1) \tag{5}$$

## Admissible Partitions

- We list the 8 partitions for $N = 8$, having vanishing coefficients, as reverse pairs

$$\{13\ 11\ 985^2 41\} \quad \{13\ 10\ 9^2 6531\} \quad Q(1)$$

$$\{13\ 11\ 9854^2 2\} \quad \{13\ 10\ 987531\} \quad Q(2)$$

$$\{13\ 11\ 976541\} \quad \{12\ 10^2 96531\} \quad Q(3)$$

$$\{12\ 11\ 97^2 4^2 1\} \quad \{12\ 10^2 7^2 532\} \quad Q(4)$$

## The q-discriminant

- Let $q\mathbf{x} = (qx_1, qx_2, \ldots, qx_N)$ and the q-discriminant of $\mathbf{x}$ be

$$D_N(q;\mathbf{x}) = \prod_{1\leq i\neq j\leq N} (x_i - qx_j) \tag{6}$$

and

$$R_N(q;\mathbf{x}) = \prod_{1\leq i\neq j\leq N} (x_i - qx_j)(qx_i - x_j) = \sum_{\lambda} c^{\lambda}(q) s_{\lambda}(\mathbf{x}) \tag{7}$$

So that

$$V_N^2(\mathbf{x}) = \prod_{1\leq i\neq j\leq N} (x_i - x_j)^2 = R_N(1;\mathbf{x}) \tag{8}$$

## q-polynomials

- Introduce q-polynomials such that

$$R_N(q;\mathbf{x}) = \sum_\lambda c^\lambda(q) s_\lambda(\mathbf{x}) \tag{9}$$

$$R_N(q;\mathbf{x}) = \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\nu \sqsubseteq (N-1)^N} \times((-q)^{|\nu|} + (-q)^{N^2-|\nu|}) s_{(N-1)^N/\nu}(\mathbf{x}) s_{\nu'}(\mathbf{x}) \tag{10}$$

- Such expansions have been evaluated as polynomials in $q$ for all admissible partitions for $N = 2, \ldots, 6$ with many examples for $N = 7, 8, 9$.

## Examples q-polynomials

| $N$ | $[c_\lambda]$ | $q(\lambda)$ | $\{\lambda\}$ |
|---|---|---|---|
| 2 | $[1]$ | $q$ | $\{2\}$ |
| | $[-3]$ | $-(q^2+q+1)$ | $\{1^2\}$ |
| 3 | $[1]$ | $q^3$ | $\{42\}$ |
| | $[-3]$ | $-q^2(q^2+q+1)$ | $\{41^2\}+\{3^2\}$ |
| | $[6]$ | $+q(q^2+1)(q^2+q+1)$ | $\{321\}$ |
| | $[-15]$ | $-(q^2+q+1)(q^4+q^2+q+1)$ | $\{2^3\}$ |

## q-polynomials for the $N = 8$ vanishing coefficients

$$-q^{17}(q^2-q+1)^2(q^2+1)^2(q^2+q+1)^5(1-q)^4 \qquad Q(1)$$

$$+q^{16}(q^2-q+1)^2(q^2+1)(q^2+q+1)^6(1-q)^4 \qquad Q(2)$$

$$+q^{16}(q^2-q+1)^2(q^2+1)^3(q^2+q+1)^5(1-q)^4 \qquad Q(3)$$

$$+q^{14}(q^2-q+1)^2(q^2+q+1)^5(1-q)^4$$

$$\times(q^{10}+q^9+3q^8+4q^6+q^5+4q^4+3q^2+q+1) \qquad Q(4)$$

Note the factor $(1-q)^4$ which vanishes for $q = 1$.

## Sum of Squares of Coefficients

- Di Francesco *et al* give the remarkable result that if

$$V^2(N) = \sum_\lambda c^\lambda s_\lambda$$

then

$$\sum_\lambda |c^\lambda|^2 = \frac{(3N)!}{N!(3!)^N} \tag{11}$$

- Can one write a similar expression for

$$\sum_\lambda |c(q)|^2?$$

## q-polynomial Sums of Squares

- Unimodal and Symmetric?

$$N = 2 \qquad q^4 + 2q^3 + 4q^2 + 2q + 1$$

$$N = 3 \qquad q^{12} + 4q^{11} + 11q^{10} + 20q^9 + 34q^8 + 44q^7$$
$$+52q^6 + 44q^5 + 34q^4 + 20q^3 + 11q^2 + 4q + 1$$

$$N = 4 \qquad q^{24} + 6q^{23} + 22q^{22} + 58q^{21} + 128q^{20} + 242q^{19}$$
$$+418q^{18} + 646q^{17} + 929q^{16} + 1210q^{15} + 1490q^{14}$$
$$+1670q^{13} + 1760q^{12} + 1670q^{11} + 1490q^{10} + 1210q^9$$
$$+646q^8 + 418q^6 + 242q^5 + 128q^4 + 58q^3 + 22q^2 + 6q + 1$$

Consider the sum $CS(N) = \sum_\lambda c^\lambda$

| N | No. Partitions | CS(N) |
|---|---|---|
| 2 | 2 | -2 |
| 3 | 5 | -14 |
| 4 | 16 | 70 |
| 5 | 59 | 910 |
| 6 | 247 | -7280 |
| 7 | 1111 | -138320 |
| 8 | 5294 | 1521520 |
| 9 | 26310 | 38038000 |
| 10 | 135281 | -532532000 |

## A Conjecture

- From the results tabulated for $N = 2, \ldots, 10$ we conjecture that

$$CS(N) = \prod_{x=0}^{[N/2]} (-3x+1) \prod_{x=0}^{[(N-1)/2]} (6x+1) \qquad (12)$$

- Exercise 1. Derive the above result.
- Exercise 2. Extend the results to the q-polynomial coefficients $c_\lambda(q)$.

## Hankel Determinants

- The Hankel matrix of order $n+1$ of a sequence $c_0, c_1, \ldots$ is the $n+1$ by $n+1$ matrix whose $(i,j)$ element is $c_{i+j}$ with $0 \leq i, j, \leq n$.
- The Hankel determinant of order $n+1$ is the determinant of the corresponding Hankel matrix,

$$det|c_{i+j}|_{0\leq i,j,\leq n} = det \begin{vmatrix} c_0 & c_1 & \ldots & c_n \\ c_1 & c_2 & \ldots & c_{n+1} \\ \vdots & \vdots & \vdots & \\ c_n & c_{n+1} & \ldots & c_{2n} \end{vmatrix} \tag{13}$$

## Hankel Determinants - Example

- Consider the sequence where $c_n = n!$. Then for $n = 3$ we have the sequence $1, 1, 2, 6, \ldots$

$$D_n^1(c) = det|c_{i+j}|_{0\leq i,j,\leq 3} = det\begin{vmatrix} 1 & 1 & 2 & 6 \\ 1 & 2 & 6 & 24 \\ 2 & 6 & 24 & 120 \\ 6 & 24 & 120 & 720 \end{vmatrix} = 144 \quad (14)$$

- *Uber eine besondere Classe der symmetrischen Determinanten*, Doctoral Thesis (1862) :- Hermann Hankel 1839-73

## Hankel HyperDeterminants

- We may define a Hankel hyperdeterminant as

$$D_n^k(c) = det_{2k}|c_{i_1+\ldots+i_{2k}}|_{0\le i_p\le n-1} \tag{15}$$

- NB. The sequence $c$ need not be restricted to just integers but may involve sequences of polynomials etc. Thus Luque and Thibon[5] consider the case where $c_n = h_n(X)$, the $n-th$ complete homogeneous symmetric function of some auxiliary variables $X = \{x_i\}$ and show that $D_n^{(k)}(h)$ may be expressed in terms of Schur functions $s_\lambda(X)$
- This problem is equivalent to determining the Schur function expansion of the even powers, $\Delta^{2k}$, of the Vandermonde determinant, $\Delta$!

## Concluding Remarks

- Remaining Problems
- *In most sciences one generation tears down what another has built, and what one has established, another undoes. In mathematics alone each generation adds a new storey to the old structure.* (Hermann Hankel 1839-1873)
- I have benefited from detailed discussions with Prof. R C King (Southampton University) and Prof. J-Y Thibon (Université de Marne-la-Vallée)
- This research has been performed under a grant from the Polish KBN (Contract No. 5P03B 057 21).