Staircase Partitions and S–functions

- Let $(\alpha) = (a, a-1, \ldots, 1)$ define a *staircase partition.* Such a partition is of *weight* $\omega_\alpha = a(a+1)/2$.
- Let $s_{a\ldots}$ designate an S–function indexed by a staircase partition whose largest part is a.
- Let $(\mu) = (\mu_1, \mu_2, \ldots)$ be a partition of weight $\omega_\mu \leq a+1$ and be the index of a S–function s_μ.
- Let $c^\gamma_{\alpha,\mu}$ be Littlewood-Richardson coefficients such that

$$s_{a\ldots} \cdot s_\mu = \sum_\gamma c^\gamma_{a\ldots,\mu} s_\gamma \tag{1}$$

1. Prove that the maximal Littlewood-Richardson coefficients are given by

$$c^{\gamma_{max}}_{a\ldots,\mu} = dim(\mu) \tag{2}$$

   where $dim(\mu)$ is evaluated for the symmetric group $\mathcal{S}_{\omega_\mu}$.

2. Prove that the number $n(\gamma_{max})$ of distinct $s_{\gamma_{max}}$ is

$$n(\gamma_{max}) = \begin{pmatrix} a+1 \\ \\ \omega_\mu \end{pmatrix} \tag{3}$$

3. Show that

$$s_{a\ldots} \cdot s_{a(a+1)/2-1\ldots} \supset dim(a\ldots) s_{a(a+1)/2\ldots} \tag{4}$$

Example

Use **SCHUR**^{TM} to show that

$$s_{7\ldots} \cdot s_{27\ldots} \supset 48,608,795,688,960 s_{28\ldots}$$

NB. Use **SCHUR**^{TM} to compute $dim(7\ldots)$ in the REP-mode for the group $S(28)$. Do not try to explicitly evaluate the S–function product!

Generalisation to Non-Staircase Partitions

The above results can be readily generalised as follows:-

- Let $(\lambda) = (\lambda_1, \lambda_2, \ldots)$ be a partition with k distinct parts and $(\mu) = (\mu_1, \mu_2, \ldots)$ be a partition of weight $\omega_\mu \leq k+1$ then

4. The maximal Littlewood-Richardson coeeficients $c_{\lambda,\mu}^{\gamma_{max}}$ are given by

$$c_{\lambda,\mu}^{\gamma_{max}} = dim(\mu) \tag{5}$$

where $dim(\mu)$ is evaluated for the symmetric group $\mathcal{S}_{\omega_\mu}$.

5. The number $n(\gamma_{max})$ of distinct $s_{\gamma_{max}}$ is

$$n(\gamma_{max}) = \begin{pmatrix} k+1 \\ \\ \omega_\mu \end{pmatrix} \tag{6}$$

Example

Using $\mathbf{SCHUR}^{TM}$ one readily finds

$$s_{32} \cdot s_{8643^221} \supset$$

$5s_{97543^21}$	$+\ 5s_{975432^2}$	$+\ 5s_{9754321^2}$	$+\ 5s_{9753^32}$
$+\ 5s_{9753^31^2}$	$+\ 5s_{97532^22^21}$	$+\ 5s_{974^23^22}$	$+\ 5s_{974^23^21^2}$
$+\ 5s_{974^232^21}$	$+\ 5s_{9743^321}$	$+\ 5s_{96543^22}$	$+\ 5s_{96543^21^2}$
$+\ 5s_{9654322^21}$	$+\ 5s_{9653^321}$	$+\ 5s_{964^23^221}$	$+\ 5s_{87543^22}$
$+\ 5s_{87543^21^2}$	$+\ 5s_{875432^21}$	$+\ 5s_{8753^321}$	$+\ 5s_{874^23^221}$
$+\ 5s_{86543^221}$			

Note that there are precisely 21 distinct partitions as predicted by Eq. (6).