Some SCHUR^{TM} Examples I : $S-$Functions

I. S-Functions: Part 1

Central to SCHUR^{TM} is the exploitation of the properties of the symmetric functions commonly known as S-functions. In the following we give some examples illustrating their properties and applications. In future pages we will illustrate other features of SCHUR^{TM}.

S-FUNCTIONS

Two useful references that describe the properties od symmetric functions, and in particula S-functions are:

[1]. I. G. Macdonald, *Symmetric Functions and Hall Polynomials 2nd edn* (Oxford: Clarendon) (1995)

[2]. B. Sagan, The Symmetric Group, Wadsworth & Brooks/Cole mathematics series, Pacific Grove, Calif. (1991).

In what follows we shall assume some familiarity with their basic properties. The S-functions are indexed by ordered partitions, (λ), of integers. In the English convention the parts of (λ) are ordered as non-increasing integers, reading from left to right. SCHUR^{TM} outputs S-functions by enclosing the partitions in curly brackets, thus the input 65543222211 is returned as {65ˆ2 432ˆ4 1ˆ2} as seen below where we mark SCHUR^{TM} input with an arrow ->.

```
(If you wish to EXIT, enter END)
(If you wish to obtain HELP, enter ?help)
DPrep Mode (with function)
DP>
->sfn
Schur Function Mode
SFN>
->65543222211
      {65^2 432^4 1^2 }
SFN>
```

Note that the S-function is returned with exponents giving the number of times a part is repeated. The partition could equally as well been input as 65ˆ2 432ˆ4 1ˆ2.

NON-STANDARD S-FUNCTIONS

S-functions indexed by partitions that are not in the standard ordering, or with some negative parts are non-standard and must be converted into standard form. This is illustrated in the following SCHUR^{TM} fragment for the non-standard partition $(2-1508\ 12\ 16)$

```
SFN>
->2~15 0 8 !12 !16
      {2-150812 16 }
SFN>
->std last
       - {10 86^3 3^2 }
SFN>
```

Note that the negative parts are input by preceding them with a tilde, $(\tilde{1}5) = (-15)$, and parts greater than 9 by an exclamation sign e.g. (!12). The command "**last**" has recalled the last set of S-functions and standardised it to produce the outputted result.

THE LITTLEWOOD-RICHARDSON RULE

Perhaps the most celebrated result in the theory of S-functions is the Littlewood-Richardson rule. If (μ) and (ν) are two partitions the S-function product $\{\mu\} \cdot \{\nu\}$ is an integral linear combination of S-functions, i.e.

$$\{\mu\} \cdot \{\nu\} = \sum_{\lambda} c^{\lambda}_{\mu\nu} \{\lambda\}$$

or equivalently

$$\{\lambda/\mu\} = \sum_{\nu} c^{\lambda}_{\mu\nu} \{\nu\}$$

The first result is often referred to as the *outer product* and the second as the *skew*. SCHUR^{TM} readily handles both outer products and skews as seen in the following fragments

```
SFN>
->o32,21
      {53} + {521} + {4^2 } + 2{431} + {42^2 } + {421^2 }
    + {3^2 2} + {3^2 1^2 } + {32^2 1}
SFN>
->sk53,21
      {41} + {32}
SFN>
->o21+3,11+2
      {5} + 3{41} + 3{32} + 3{31^2 } + 2{2^2 1} + {21^3 }
SFN>
```

The last result shows SCHUR^{TM} handling lists of S-functions. Commands may be nested as seen in this fragment

```
SFN>
->o21,sk53,21
      {62} + {61^2 } + 2{53} + 3{521} + {51^3 } + {4^2 } + 3{431}
 + 2{42^2 } + 2{421^2 } + {3^2 2} + {3^2 1^2 } + {32^2 1}
SFN>
```

EXERCISES I

Using SCHUR^{TM} can lead to many conjectures and resultant exercises. In this exercise we consider two S-functions $\{a^n\}$ and $\{b^n\}$ whose partitions (a^n) and (b^n) both have the shape of a rectangle as seen by the example of $(a^3) = (2^3)$ and $(b^3) = (4^3)$ depicted below

```
SFN>
->young 2^3 + 4^3
  OO   OOOO
  OO   OOOO
  OO   OOOO
SFN>
```

Let us now form the outer product $\{2^3\} \cdot \{4^3\}$

```
SFN>
->o222,444
      {6^3 } + {6^2 51} + {6^2 42} + {65^2 1^2 } + {65421} + {64^2 2^2 }
    + {5^3 1^3 } + {5^2 421^2 } + {54^2 2^2 1} + {4^3 2^3 }
SFN>
```

Notice that the resulting S-functions all occur with multiplicity 1. This suggests the following exercise.

EXERCISE 1

(i) Is the outer product $\{a^n\} \cdot \{b^n\}$ multiplicity free for all $[a, b, n]$?

(ii) Is the outer product $\{a^m\} \cdot \{b^n\}$ multiplicity free for all $[a, b, m, n]$?

Now restrict the above outer product to resultant partitions of length 5 or less and display their Young frames as below

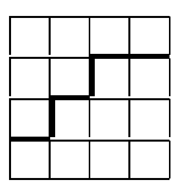
```
SFN>
->len5last
     {6^3 } + {6^2 51} + {6^2 42} + {65^2 1^2 } + {65421} + {64^2 2^2 }
SFN>
->young last
  000000   000000   000000   000000   000000   000000
  000000   000000   000000   00000    00000    0000
  000000   00000    0000     00000    0000     0000
           0        00       0        00       00
                             0        0        00
SFN>
```

Let us now extend the above frames by attaching x's to make each frame into a 6×6 rectangle as below

```
  000000   000000   000000   000000   000000   000000
  000000   000000   000000   00000x   00000x   0000xx
  000000   00000x   0000xx   00000x   0000xx   0000xx
  xxxxxx   0xxxxx   00xxxx   0xxxxx   00xxxx   00xxxx
  xxxxxx   xxxxxx   xxxxxx   0xxxxx   0xxxxx   00xxxx
  xxxxxx   xxxxxx   xxxxxx   xxxxxx   xxxxxx   xxxxxx
```

Note that vif the shapes formed by the x's are rotated by 180° we see that they are the same shape as those formed by the circles '0'. Such shapes are said to be *self-complementary* and are defined for a particular value of n, in the above case $n = 6$.

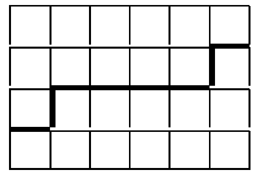
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be an ordered partition such that $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 0)$, possibly with trailing zeros to make the total number of parts of λ equal to a positive integer n. Such a partition may be inscribed in a box $B = (\lambda_1^n)$ having λ_1 columns and n rows as illustrated for the particular case of the partition (4210).



The cells in the lower portion of B not occupied by those of λ describe the shape of a partition λ^c (after rotation by π) which we shall term the *complement* of λ where

$$\lambda^c = (\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, 0) \tag{1}$$

A partition λ^{sc} will be said to *self-complementary* if $\lambda = \lambda^c$. In that case the box B involves two equal parts (to within a rotation by π) as shown below for the partition (6510).



EXERCISE 2

Show that for the S-function product $\{a^m\} \cdot \{b^m\}$ the resultant S-functions of length $2m-1$, or less, are self-complementary for all $[a, b, m]$.

EXERCISE 3

Show that for any non-trivial partition (λ) that $\{\lambda/1\} \cdot \{1\}$ contains the S-function $\{\lambda\}$ with a multiplicity equal to the number of distinct parts of $\{\lambda\}$.

It is a simple matter to write a function in SCHUR^{TM} to test Exercise 3 for particular partitions $(\lambda\}$ as seen in the following SCHUR^{TM} fragment

```
DPrep Mode (with function)
DP>
->setfn1
=-
->sfn
=-
->enter sv1
=-
->o sk sv1,1,1
=-
->stop
```

`DP>` Notice the use of a sequence of commands in a single line.

As an example of running the function we have

```
->fn1
Schur Function Mode
enter sv1
->4321
      {532} + {531^2 } + {52^2 1} + {4^2 2} + {4^2 1^2 } + {43^2 }
   + 4{4321} + {431^3 } + {42^3 } + {42^2 1^2 } + {3^3 1} + {3^2 2^2 }
   + {3^2 21^2 }
```

`SFN>` Note that the partition (4321) involves 4 distinct parts which is indeed the multiplicity of $\{4321\}$ as conjectured.

EXERCISE 4

The above SCHUR^{TM} fragment suggests that $\{\lambda/1\} \cdot \{1\}$ is multiplicity free apart from that of $\{\lambda\}$. Is this generally the case?

FURTHER INFORMATION

The above exercises show how use of SCHUR^{TM} can lead to conjectures. The proofs of the conjectures are outlined in

M. Yang and B. G. Wybourne, *Squares of S-functions of special shapes*, J. Phys. A: Math. Gen. **28**, 7011-7017 (1995).

and references therein.

BACK TO THE FUTURE

This is the first set of SCHUR^{TM} exercises and examples. We will illustrate further examples of using SCHUR^{TM} to study symmetric functions and look at their applications to some problems in physics. Return to this page in April for the next instalment. We end with an unanswered problem. If you solve it let me know.

What is the Maximum Littlewood-Richardson Coefficient?
for the Product of Two Staircase S-functions?

Suppose that λ and μ are two staircase partitions. Under what conditions does the maximal Littlewood-Richardson coefficient involve just one partition ν? Below are some typical results deduced using SCHUR^{TM}.

$\{\lambda\}\cdot\{\mu\}$	Maximum $c^{\nu}_{\lambda\mu}$	S-functions
$\{21\}\cdot\{21\}$	2	$\{321\}$
$\{4321\}\cdot\{321\}$	8	$\{64321\}+\{54321^2\}$
$\{4321\}\cdot\{4321\}$	18	$\{65431^2\}+\{6542^2\,1\}+\{653^2\,21\}+\{64^2\,321\}$
$\{54321\}\cdot\{321\}$	16	$\{654321\}$
$\{54321\}\cdot\{4321\}$	40	$\{765321^2\}+\{76432^2\,1\}+\{7543^2\,21\}$
$\{54321\}\cdot\{54321\}$	176	$\{8654321^2\}$
$\{654321\}\cdot\{4321\}$	88	$\{8754321^2\}+\{865432^2\,1\}$
$\{654321\}\cdot\{54321\}$	640	$\{87654321\}$
$\{654321\}\cdot\{654321\}$	2064	$\{987543^2\,21\}$
$\{7654321\}\cdot\{4321\}$	192	$\{97654321^2\}$
$\{7654321\}\cdot\{54321\}$	1160	$\{987643^2\,21\}+\{98754^2\,321\}$
$\{7654321\}\cdot\{654321\}$	10 128	$\{10\ 9765432^2\,1\}$

I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave to others the pleasure of discovery
René Descartes (1596 - 1650) *La Geometrie*