Aust. J. Phys., 1978, **31**, 117–29

S-wave Colour Singlet States of Multiquark Configurations

B. G. Wybourne

Department of Physics, University of Canterbury, Christchurch 1, New Zealand.

Abstract

Simple methods for determining and systematically labelling the S-wave colour singlet states of multiquark configurations are developed and results are given for the quark configurations Q^N ($N = 0 \pmod 3$) and for the quark–antiquark configurations $Q^2\bar{Q}^2$, $Q^4\bar{Q}$ and $Q^3\bar{Q}^3$. The masses of the colour singlet states associated with the Q^6 and Q^9 configurations are calculated using the MIT bag model and a number of conclusions concerning these mass spectra are drawn.

Introduction

An important part of current theories of quantum chromodynamics (QCD) is the hypothesis that the only observable states are those corresponding to colour singlets under the colour group SU_3^C. In QCD the hadrons are described in terms of confined coloured quarks of various flavours and massless vector gluons. The MIT bag model of hadrons (Chodos *et al.* 1974*a*, 1974*b*; DeGrand *et al.* 1975; DeGrand and Jaffe 1976) has provided a practical model for giving a phenomenological description of hadrons within the framework of QCD. Jaffe (1977*a*, 1977*b*, 1977*c*) has recently endeavoured to extend the bag model to the description of the S-wave colour-singlet multiquark hadrons.

In this paper we describe some simple methods for determining the colour singlets that arise in multiquark configurations and give a systematic method of labelling the S-wave colour singlet states. Results are given for the quark configurations Q^N ($N = 0 \pmod 3$) and for the quark–antiquark configurations $Q^2\bar{Q}^2$, $Q^4\bar{Q}$ and $Q^3\bar{Q}^3$. A group-theoretical description of multiquark hadrons is developed and the matrix elements of the gluon interaction term are calculated for the Q^N configurations. The masses of the states associated with the Q^6 and Q^9 S-wave colour singlets are then calculated on the basis of the MIT bag model. A number of conclusions concerning these mass spectra are drawn.

Special Unitary Groups SU_n

The special unitary groups SU_n play a fundamental role in the classification of multiquark hadron states. Here we label the irreducible representations (irreps) of SU_n by ordered partitions (λ) of integers into not more than $n-1$ parts with the understanding that the irreps corresponding to n-part partitions are equivalent to irreps corresponding to fewer than n parts via the equivalence

$$\{\lambda_1, \lambda_2, ..., \lambda_n\} \equiv \{\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, ..., 0\}.$$

The partitions

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \qquad \text{and} \qquad (\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots)$$

are said to correspond to irreps of SU_n that are *contragredient* to one another. If the equivalence

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \equiv (\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots)$$

holds then the corresponding irreps are said to be *self-contragredient* (Wybourne and Bowick 1977).

Antisymmetric Multiquark Hadron States

The quarks are fermions and as such multiquark states must be totally antisymmetrized. However, the colour hypothesis gives a stronger condition: not only must the states be antisymmetric but they must also be colour singlets if they are to correspond to observable states.

For maximal generality we shall at this stage assume N flavours of quarks are associated with a flavour group SU_N^{fl} and that the basic set of coloured quarks spans the $\{1\}\{1\}'$ irrep of the direct product group $SU_N^{\mathrm{fl}} \times SU_6^{CS}$, where SU_6^{CS} is the colour–spin group. The coloured antiquarks will span the contragredient irrep $\{1^{N-1}\}\{1^5\}'$ of $SU_N^{\mathrm{fl}} \times SU_6^{CS}$.

The direct product group $SU_N^{\mathrm{fl}} \times SU_6^{CS}$ may be embedded in the larger group SU_{6N}, with the coloured quarks spanning the $\{1\}$ irrep and the antiquarks the $\{1^{6N-1}\}$ irrep. The antisymmetric S-wave states of the quark configuration Q^x will span the $\{1^x\}$ irrep and those of the antiquark configuration $\bar{Q}^x$ the contragredient $\{1^{6N-x}\}$ irrep of SU_{6N}.

The irreps of SU_{6N} may be decomposed into those of $SU_N^{\mathrm{fl}} \times SU_6^{CS}$ by noting the S-function result (King 1975)

$$\{\lambda\} \to \sum_{\xi} (\{\lambda\} \circ \{\xi\}) \{\xi\}, \tag{1}$$

where the symbol $\circ$ denotes the inner S-function multiplication (Wybourne 1970, 1974). For the special case of the $\{1^n\}$ irreps we have the simpler result

$$\{1^n\} \to \sum_{\xi} \{\tilde{\xi}\}\{\xi\}, \tag{2}$$

where $\{\xi\}$ is an irrep of SU_6^{CS} involving a partition (ξ) of n into not more than six parts and $\{\tilde{\xi}\}$ is the irrep of SU_N^{fl} involving a partition of n into not more than N parts and conjugate to (ξ). The tables of Wybourne (1970) permit rapid evaluation of these branching rules.

In the cases of present interest it suffices to identify SU_N^{fl} with SU_3^{fl}, though there is no difficulty in extending the classification of the multiquark hadrons for larger flavour groups. The branching rules for $SU_{18} \to SU_3^{\mathrm{fl}} \times SU_6^{CS}$ are given for the irreps $\{1^x\}$ $(x = 0, 1, \ldots, 9)$ in Table 1. The branching rules for the irreps $\{1^{18-x}\}$ may be obtained from those for $\{1^x\}$ by replacing the $SU_3^{\mathrm{fl}} \times SU_6^{CS}$ irreps by their contragredient partners.

As it stands, Table 1 gives a classification of the antisymmetric states of the multiquark configurations Q^x and $\bar{Q}^x$. However, the colour hypothesis is stronger than just antisymmetrization and we should exclude from the Q^x and $\bar{Q}^x$ antisymmetric states all those states that do not transform as colour singlets. This means we must

Table 1. Branching rules for $SU_{18} \rightarrow SU_3^{fl} \times SU_6^{CS}$

Dimensions $D_{(\lambda)}$	Irreps for SU_{18}	Branching to $SU_3^{fl} \times SU_6^{CS}$
1	$\{0\}$	$\{0\}\{0\}$
18	$\{1\}$	$\{1\}\{1\}$
153	$\{1^2\}$	$\{2\}\{1^2\}+\{1^2\}\{2\}$
816	$\{1^3\}$	$\{21\}\{21\}+\{3\}\{1^3\}+\{0\}\{3\}$
3 060	$\{1^4\}$	$\{4\}\{1^4\}+\{31\}\{21^2\}+\{1\}\{31\}+\{2^2\}\{2^2\}$
8 568	$\{1^5\}$	$\{5\}\{1^5\}+\{41\}\{21^3\}+\{32\}\{2^21\}+\{1^2\}\{32\}+\{2\}\{31^2\}$
18 564	$\{1^6\}$	$\{6\}\{0\}+\{51\}\{21^4\}+\{42\}\{2^21^2\}+\{3\}\{31^3\}+\{3^2\}\{2^3\}+\{21\}\{321\}$ $+\{0\}\{3^2\}$
31 824	$\{1^7\}$	$\{61\}\{1\}+\{52\}\{2^21^3\}+\{4\}\{31^4\}+\{43\}\{2^31\}+\{31\}\{321^2\}+\{2^2\}\{32^2\}$ $+\{1\}\{3^21\}$
43 758	$\{1^8\}$	$\{62\}\{1^2\}+\{5\}\{2\}+\{53\}\{2^31^2\}+\{41\}\{321^3\}+\{4^2\}\{2^4\}+\{32\}\{32^21\}$ $+\{2\}\{3^21^2\}+\{1^2\}\{3^22\}$
48 620	$\{1^9\}$	$\{63\}\{1^3\}+\{54\}\{2^41\}+\{51\}\{21\}+\{42\}\{32^21^2\}+\{3\}\{3^21^3\}+\{3^2\}\{32^3\}$ $+\{21\}\{3^221\}+\{0\}\{3^3\}$

Table 2. S-wave colour singlet states in Q^{3x} quark configurations

Quark configuration	SU_3^{fl} irrep	SU_6^{CS} irrep	Spin S	Casimir operator C_6	Ratio $\Delta/\bar{M}$	M^ (MeV)
Q^3	$\{3\}$	$\{1^3\}$	$\frac{3}{2}$	42	8	1237
	$\{21\}$	$\{21\}$	$\frac{1}{2}$	66	-8	941
Q^6	$\{6\}$	$\{0\}$	0	0	48	2799
	$\{51\}$	$\{21^4\}$	1	48	80/3	2507
	$\{42\}$	$\{2^21^2\}$	2, 0	80	8	2242
	$\{3^2\}$	$\{2^3\}$	3, 1	96	8/3	2164
	$\{3\}$	$\{31^3\}$	1	96	8/3	2164
	$\{21\}$	$\{321\}$	2, 1	120	$-28/3$	1986
	$\{0\}$	$\{3^2\}$	0	144	-24	1761
Q^9	$\{63\}$	$\{1^3\}$	$\frac{3}{2}$	42	56	3715
	$\{54\}$	$\{2^41\}$	$\frac{1}{2}$	66	40	3518
	$\{51\}$	$\{21\}$	$\frac{1}{2}$	66	40	3518
	$\{42\}$	$\{32^21^2\}$	$\frac{5}{2}, \frac{3}{2}, \frac{1}{2}$	98	24	3321
	$\{3^2\}$	$\{32^3\}$	$\frac{3}{2}$	114	20	3266
	$\{3\}$	$\{3^21^3\}$	$\frac{3}{2}$	114	20	3266
	$\{21\}$	$\{3^221\}$	$\frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$	138	4	3060
	$\{0\}$	$\{3^3\}$	$\frac{9}{2}, \frac{5}{2}, \frac{3}{2}$	162	-4	2954

^ These masses are computed assuming $m_s = 0$ and are given for the smallest values of S for a given colour–spin multiplet.

immediately exclude all configurations of nonzero triality and then examine the $SU_6^{CS} \rightarrow SU_2^S \times SU_3^C$ decompositions—discarding all except the colour singlet states. These branching rules again follow from equation (1) where now the S-functions arising in $\{\lambda\} \circ \{\xi\}$ involve partitions into not more than two parts and those of $\{\xi\}$ not more than three parts. The spin S associated with a given irrep $\{\lambda_1, \lambda_2\}$ of SU_2^S is deduced by noting the equivalence

$$\{\lambda_1, \lambda_2\} \equiv \{\lambda_1 - \lambda_2, 0\}$$

and that

$$2S = \lambda_1 - \lambda_2 \,. \tag{3}$$

The S-wave colour singlet states in the Q^{3x} quark configurations for $x = 1,2,3$ are given in Table 2 along with the eigenvalues of the quadratic Casimir operator C_6 for the appropriate SU_6^{CS} irreps. The Casimir operator C_6 has been defined to agree with that used by Jaffe (1977*b*), though the eigenvalues were more readily deduced using the standard result (Judd 1963) for an n-part partition (λ) of weight m corresponding to the irrep $\{\lambda\}$ of SU_6^{CS} to give

$$3C_6\{\lambda\} = 12\sum_{i=1}^{n}\lambda_i(\lambda_i+7-2i)-2m^2. \tag{4}$$

The eigenvalues of the operator

$$\Delta = 8N-\tfrac{1}{2}C_6(\{\mu\}^{CS})+\tfrac{4}{3}S(S+1) \tag{5}$$

are given in the penultimate column of Table 2. The eigenvalues of Δ for Q^{12} may be found by adding 48 to the corresponding eigenvalues found for Q^6.

Colour Singlet States for $Q^x\bar{Q}^y$ Configurations

Table 1 provides a basis for obtaining antisymmetric states of the multiquark hadron configurations $Q^x\bar{Q}^y$ $(x-y = 0 \pmod 3)$ and hence their colour singlet states. Antisymmetrization need only be carried out explicitly within the configurations Q^x and $\bar{Q}^y$. There remains the specification of the coupling between the states of these two configurations to form the final states of the composite. While it is possible to construct a grandiose scheme where all the S-wave states of all possible $Q^x\bar{Q}^y$ configurations span a single irrep and then reduce this irrep through a chain of subgroups exploiting quasi-spin type concepts etc., it is probably best to use a scheme that does not obscure the genealogical origin of the final states too much. With this in mind we shall use the group scheme

$$\begin{aligned} SU_{18}\times SU_{18} &\rightarrow (SU_3^{\rm fl}\times SU_6^{CS})\times(SU_3^{\rm fl}\times SU_6^{CS}) \\ &\rightarrow SU_3^{\rm fl}\times SU_6^{CS}\rightarrow SU_2^S\times SU_3^{\rm fl}\times SU_3^C. \end{aligned} \tag{6}$$

The antisymmetric states of the $Q^x\bar{Q}^y$ configuration will span the $\{1^x\}\{1^{18-y}\}$ irrep of $SU_{18}\times SU_{18}$. The reduction $SU_{18}\rightarrow SU_3^{\rm fl}\times SU_6^{CS}$ is already accomplished in Table 1. The reduction of

$$(SU_3^{\rm fl}\times SU_6^{CS})\times(SU_3^{\rm fl}\times SU_6^{CS})\rightarrow SU_3^{\rm fl}\times SU_6^{CS}$$

follows from the evaluation of the appropriate S-function outer products (Wybourne 1970) to give the results shown in Table 3 relevant to the classification of the states of the quark configurations $Q^x\bar{Q}^y$. The classification of colour singlet states is completed by identifying those irreps of SU_6^{CS} that contain an SU_3^C singlet under the reduction $SU_6^{CS}\rightarrow SU_2^S\times SU_3^C$. The values of the spin S associated with the colour singlets can be found from the results given in Table 2 together with the additional results:

$$\{421^3\}\supset S=\tfrac{3}{2},\tfrac{1}{2}; \qquad \{42^4\}\supset S=2,0;$$
$$\{52^31\}\supset S=2,1; \qquad \{63^4\}\supset S=3,1.$$
$$\{432^21\}\supset S=3,2,1(2),0;$$

The spins S associated with the colour singlets of the above SU_6^{CS} irreps were determined by a method based on equation (1) together with equation (47) of Wybourne (1970). In this way it was possible to arrive at the values of S without decomposing irreps of SU_6^{CS} completely.

Table 3. Classification of states of quark configurations $Q^x\bar{Q}^y$

Config.	$(SU_3^{\mathrm{fl}} \times SU_6^{CS}) \times (SU_3^{\mathrm{fl}} \times SU_6^{CS})$	$SU_3^{\mathrm{fl}} \times SU_6^{CS}$
$Q\bar{Q}$	$(\{1\}\{1\}) \times (\{1^2\}\{1^5\})$	$[\{0\}+\{21\}] \times [\{0\}+\{21^4\}]$
$Q^2\bar{Q}^2$	$(\{2\}\{1^2\}) \times (\{2^2\}\{1^4\})$	$[\{0\}+\{21\}+\{42\}] \times [\{0\}+\{21^4\}+\{2^21^2\}]$
	$(\{2\}\{1^2\}) \times (\{1\}\{2^5\})$	$[\{21\}+\{3\}] \times [\{21^4\}+\{3^22^3\}]$
	$(\{1^2\}\{2\}) \times (\{2^2\}\{1^4\})$	$[\{21\}+\{3^2\}] \times [\{21^4\}+\{31^3\}]$
	$(\{1^2\}\{2\}) \times (\{1\}\{2^5\})$	$[\{0\}+\{21\}] \times [\{0\}+\{21^4\}+\{42^4\}]$
$Q^4\bar{Q}$	$(\{4\}\{1^4\}) \times (\{1^2\}\{1^5\})$	$[\{3\}+\{51\}] \times [\{1^3\}+\{2^41\}]$
	$(\{31\}\{21^2\}) \times (\{1^2\}\{1^5\})$	$[\{21\}+\{3\}+\{42\}] \times [\{1^3\}+\{21\}+\{32^21^2\}]$
	$(\{1\}\{31\}) \times (\{1^2\}\{1^5\})$	$[\{0\}+\{21\}] \times [\{21\}+\{3\}+\{421^3\}]$
	$(\{2^2\}\{2^2\}) \times (\{1^2\}\{1^5\}$	$[\{21\}+\{3^2\}] \times [\{21\}\{3^21^3\}]$
$Q^3\bar{Q}^3$	$(\{21\}\{21\}) \times (\{21\}\{2^41\})$	$[\{0\}+2\{21\}+\{3\}+\{3^2\}+\{42\}] \times [\{0\}+2\{21^4\}$ $+\{2^21^2\}+\{31^3\}+\{3^22^3\}+\{42^4\}+\{432^21\}]$
	$(\{21\}\{21\}) \times (\{3^2\}\{1^3\})$	$[\{21\}+\{3^2\}+\{42\}+\{54\}]$ $\times [\{21^4\}+\{2^21^2\}+\{31^3\}+\{321\}]$
	$(\{21\}\{21\}) \times (\{0\}\{3^5\})$	$[\{21\}] \times [\{21^4\}+\{3^22^3\}+\{42^4\}+\{543^3\}]$
	$(\{3\}\{1^3\}) \times (\{21\}\{2^41\})$	$[\{21\}+\{3\}+\{42\}+\{51\}]$ $\times [\{21^4\}+\{2^21^2\}+\{3^22^3\}+\{3^321\}]$
	$(\{3\}\{1^3\}) \times (\{3^2\}\{1^3\})$	$[\{0\}+\{21\}+\{42\}+\{63\}]$ $\times [\{0\}+\{21^4\}+\{2^21^2\}+\{2^3\}]$
	$(\{3\}\{1^3\}) \times (\{0\}\{3^5\})$	$[\{3\}] \times [\{3^22^3\}+\{4^33^2\}]$
	$(\{0\}\{3\}) \times (\{21\}\{2^41\})$	$[\{21\}] \times [\{21^4\}+\{31^3\}+\{42^4\}+\{52^31\}]$
	$(\{0\}\{3\}) \times (\{3^2\}\{1^3\})$	$[\{3^2\}] \times [\{31^3\}+\{41^2\}]$
	$(\{0\}\{3\}) \times (\{0\}\{3^5\})$	$[\{0\}] \times [\{0\}+\{21^4\}+\{42^4\}+\{63^4\}]$

Exotic Multiquark Hadron Flavour Multiplets

The commonly observed hadrons, the mesons and baryons, are associated with the triality zero irreps $\{0\}$, $\{21\}$, $\{3\}$ and $\{3^2\}$ of the flavour group SU_3^{fl}. Inspection of Tables 2 and 3 shows that the multiquark hadrons can lead to exotic triality zero irreps of SU_3^{fl}. The isospin–hypercharge (I, Y) content of these irreps may be found by noting that if $\{\lambda\}$ is an irrep of SU_3^{fl} of weight m then under $SU_3^{\mathrm{fl}} \to SU_2^I \times U_1^Y$ we have

$$\{\lambda\} \to \sum_n \{\lambda/n\}\{\tfrac{1}{3}m-n\}, \tag{7}$$

where n is any integer compatible with the S-function division and $\{\lambda/n\}$ labels the irreps of the isospin group SU_2^I involving partitions (μ) into at most two parts while $\{\frac{1}{3}m-n\}$ labels the irreps of the hypercharge group U_1^Y. Specifically,

$$2I = \mu_1 - \mu_2 \qquad \text{and} \qquad Y = \tfrac{1}{3}m - n. \tag{8}$$

The values of (I, Y) associated with the exotic irreps of SU_3^{fl} that arise in Tables 2 and 3 are given in Table 4. The eigenvalues of the SU_3 quadratic Casimir operator may be readily evaluated by noting that

$$3C_3\{\mu_1\mu_2\} = 4[\mu_1^2+\mu_2^2-\mu_1\mu_2+3\mu_1], \tag{9}$$

where we have chosen our normalization to agree with that used by Jaffe (1977*b*).

Table 4. Exotic $SU_3^{\mathrm{fl}} \to SU_2^I \times U_1^Y$ decompositions

$D_{(\lambda)}$	SU_3^{fl}	Isospin–hypercharge (I, Y)
27	{42}	$(1,2)+(\frac{3}{2},1)+(\frac{1}{2},1)+(2,0)+(1,0)+(0,0)+(\frac{3}{2},-1)+(\frac{1}{2},-1)+(1,-2)$
35	{51}	$(2,2)+(\frac{5}{2},1)+(\frac{3}{2},1)+(2,0)+(1,0)+(\frac{3}{2},-1)+(\frac{1}{2},-1)+(1,-2)$ $+(0,-2)+(\frac{1}{2},-3)$
28	{6}	$(3,2)+(\frac{5}{2},1)+(2,0)+(\frac{3}{2},-1)+(1,-2)+(\frac{1}{2},-3)+(0,-4)$
64	{63}	$(\frac{3}{2},3)+(2,2)+(1,2)+(\frac{5}{2},1)+(\frac{3}{2},1)+(\frac{1}{2},1)+(3,0)+(2,0)+(1,0)$ $+(0,0)+(\frac{5}{2},-1)+(\frac{3}{2},-1)+(\frac{1}{2},-1)+(2,-2)+(1,-2)+(\frac{3}{2},-3)$

Multiquark Hadron States

The colour singlet S-wave states of the quark configuration Q^N with $N = 0$ (mod 3) follow directly from Tables 1 and 4 with a typical state being written as

$$|\{1^N\}\,\{\lambda\}^{\mathrm{fl}}\,\{\mu\}^{CS}\,\{0\}^C\,SS_z\,II_z\,Y\rangle, \tag{10}$$

where the trivial colour hypercharge and isospin numbers are suppressed.

The corresponding states of the quark–antiquark configuration $Q^x\bar{Q}^y$ with $x-y = 0$ (mod 3) follow from Tables 3 and 4 with a typical state being written as

$$|\{1^N\}\,\{\lambda_1\}_Q^{\mathrm{fl}}\,\{\mu_1\}_Q^{CS}\,\{\lambda_2\}_{\bar{Q}}^{\mathrm{fl}}\,\{\mu_2\}_{\bar{Q}}^{CS};\,r_{\mathrm{f}}\{\lambda_{12}\}^{\mathrm{fl}}\,r_{\mathrm{c}}\{\mu_{12}\}^{CS}\,\{0\}^C\,SS_z\,II_z\,Y\rangle, \tag{11}$$

where r_{f} and r_{c} are multiplicity indexes.

Diagonalization of Colour–Spin Operator

Jaffe (1977*b*) has shown that the diagonalization of the quark–gluon interaction for multiquark hadrons involves the evaluation of the matrix elements of the two-particle colour–spin operator

$$O_{\mathrm{g}} = -\sum_{a=1}^{8}\sum_{i<j}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\,\lambda_i^a\,\lambda_j^a, \tag{12}$$

where $\boldsymbol{\sigma}_i$ and λ_i^a are the spin and colour vectors for the ith quark. The operator O_{g} acts only on the colour–spin group SU_6^{CS} and its subgroups and hence we can effectively suppress the flavour group representations and use the abbreviated kets

$$|N\alpha\,\{\mu\}^{CS}\,\{0\}^C\,SS_z\rangle \tag{13}$$

for Q^N and

$$|N\alpha\,\{\mu_1\}_Q^{CS}\,\{\mu_2\}_{\bar{Q}}^{CS};\,r\,\{\mu_{12}\}^{CS}\,\{0\}^C\,SS_z\rangle \tag{14}$$

for $Q^x\bar{Q}^y$, with α denoting the associated sets of flavour representations and r being a multiplicity index.

If O_g acts on the ket (13) we obtain the eigenvalues given by equation (5) above, which are already listed in Table 2. The operation of O_g on the ket (14) yields both diagonal and off-diagonal elements, and in this case it can be useful to partially decouple (14) to give

$$
\begin{aligned}
&| N\alpha \{\mu_1\}_Q^{CS} \{\mu_2\}_{\bar{Q}}^{CS}; r_c\{\mu_{12}\}^{CS} \{0\}^C SS_z\rangle \\
&\quad = \sum_{\pi_1,S_1,S_2} \langle \{\mu_1\}_Q^{CS} \{\pi_1\}^C S_1, \{\mu_2\}_{\bar{Q}}^{CS} \{\pi_1\}^{*C} S_2 \,|\, r_c\{\mu_{12}\}^{CS} \{0\}^C S\rangle \\
&\qquad\qquad \times | N\alpha \{\mu_1\}_Q^{CS} \{\pi_1\}^C S_1 \{\mu_2\}_{\bar{Q}}^{CS} \{\pi_1\}^{*C} S_2; \{0\}^C SS_z\rangle ,
\end{aligned}
\tag{15}
$$

where $\{\pi_1\}^*$ is contragredient to $\{\pi_1\}$. The matrix elements of O_g can then be written as

$$
\begin{aligned}
&\langle N\alpha \{\mu_1\}_Q^{CS} \{\mu_2\}_{\bar{Q}}^{CS}; r_c\{\mu_{12}\}^{CS} \{0\}^C SS_z \,|\, O_g \,|\, N\alpha' \{\mu_1'\}_Q^{CS} \{\mu_2'\}_{\bar{Q}}^{CS}; r_c'\{\mu_{12}'\}^{CS} \{0\}^C S'S_z'\rangle \\
&\quad = \delta_{\alpha\alpha'}\, \delta_{SS'}\, \delta_{S_zS_z'}\, \delta_{\mu_1\mu_1'}\, \delta_{\mu_2\mu_2'} \\
&\qquad \times \Big\{ \delta_{\mu_{12}\mu_{12}'} [8N + \tfrac{1}{2} C_6(\{\mu_{12}\}) - C_6(\{\mu_1\}) - C_6(\{\mu_2\}) - \tfrac{4}{3} S(S+1)] \\
&\qquad\quad + 2 \sum_{S_1,S_2,\pi_1} \langle r_c\{\mu_{12}\} \{0\} S \,|\, \{\mu_1\} \{\pi_1\} S_1 \{\mu_2\} \{\pi_1\}^* S_2\rangle \\
&\qquad\qquad\qquad \times \langle \{\mu_1\} \{\pi_1\} S_1 \{\mu_2\} \{\pi_1\}^* S_2 \,|\, r_c'\{\mu_{12}\} \{0\} S\rangle \\
&\qquad\qquad\qquad \times [C_3(\{\pi_1\}) + \tfrac{4}{3}\{S_1(S_1+1) + S_2(S_2+1)\}] \Big\} .
\end{aligned}
\tag{16}
$$

In the special case of the configurations $Q^{N-1}\bar{Q}$ the result (16) simplifies to

$$
\begin{aligned}
&\langle N\alpha \{\mu_1\}_Q^{CS} \{1^5\}_{\bar{Q}}^{CS}; \{\mu_{12}\}^{CS} \{0\}^C SS_z \,|\, O_g \,|\, N\alpha \{\mu_1\}_Q^{CS} \{1^5\}_{\bar{Q}}^{CS}; \{\mu_{12}'\}^{CS} \{0\}^C SS_z\rangle \\
&\quad = \delta_{\mu_{12}\mu_{12}'} [8N - \tfrac{32}{3} + \tfrac{1}{2} C_6(\{\mu_{12}\}) - C_6(\{\mu_1\}) - \tfrac{4}{3} S(S+1)] \\
&\qquad + \tfrac{8}{3} \sum_{S_1} S_1(S_1+1) \langle \{\mu_{12}\} \{0\} S \,|\, \{\mu_1\} \{1\} S_1 \{1^5\} \{1^2\} \tfrac{1}{2}\rangle \\
&\qquad\qquad \times \langle \{\mu_1\} \{1\} S_1 \{1^5\} \{1^2\} \tfrac{1}{2} \,|\, \{\mu_{12}'\} \{0\} S\rangle .
\end{aligned}
\tag{17}
$$

The $SU_6 \supset SU_2 \times SU_3$ isoscalar factors may be expressed in terms of $3jm$ factors (Butler 1975); the relevant $3jm$ factors are all primitive (Butler and Wybourne 1976) and may be readily evaluated. Equation (17) involves two terms: a diagonal term that is dominant in all cases of interest and nondiagonal terms that require the use of equations (8) together with the appropriate isoscalar factors.

Multiquark MIT Bag Model Mass Calculations

In the MIT bag model the quark and gluon fields are assumed to be permanently confined to a spherical bag of radius R (we ignore questions of bag deformation) and the energy $E(R)$ of the S-wave hadrons is a function of the bag radius. This radius is fixed by the boundary condition

$$(\partial E/\partial R)\,|_{R=R_0} = 0 , \tag{18}$$

which amounts to balancing the field pressures with the bag-confining pressure B.

The energy of the hadrons contains four terms:

$$E(R) = E_B + E_0 + E_K + E_g. \tag{19}$$

The first term

$$E_B = \tfrac{4}{3}\pi B R^3 \tag{20}$$

is the energy associated with the confining pressure B, while the second term

$$E_0 = -Z_0/R \tag{21}$$

gives a phenomenological estimate of the effects of zero-point fluctuations of the quark and gluon fields confined to a sphere of radius R.

The third term in equation (19) is the quark kinetic energy

$$E_K = \sum_{i=1}^{N} \omega_i(m_i R)/R, \tag{22}$$

where

$$\omega_i(m_i R) = \left(x(m_i R)^2 + m_i^2 R^2\right)^{\frac{1}{2}} \tag{23}$$

with $x(m_i R)$ being the dimensionless wave number of an S-wave quark of mass m_i in a cavity of radius R. The quantity $x(m_i R)$ as a function of $m_i R$ has been evaluated by DeGrand *et al.* (1975). For a state involving n_0 non-strange quarks of equal mass m_0 and n_s strange quarks of mass m_s we may replace equation (22) by

$$E_K = \left(n_0\, \omega(m_0 R) + n_s\, \omega(m_s R)\right)/R. \tag{24}$$

The fourth term in equation (19)

$$E_g = -(\alpha_c/R) \sum_{a=1}^{8} \sum_{i<j} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\, \lambda_i^a \lambda_j^a\, M(m_i R, m_j R) \tag{25}$$

represents the quark–gluon interaction to first-order in the quark–gluon coupling constant α_c. The integral $M(m_i R, m_j R)$ has been evaluated and plotted as a function of mR by DeGrand *et al.* (1975). The calculation of the gluon interaction matrix elements can be enormously simplified for an N-quark state involving n_s strange quarks by making the replacement (Jaffe 1977*a*, 1977*b*)

$$M(m_i R, m_j R) \to M\left((n_s/N)m_s R, (n_s/N)m_s R\right). \tag{26}$$

For $n_s = 0$ or N the replacement is exact while for the other values of n_s it is equivalent to making a linear interpolation. The introduction of this approximation amounts to ignoring splittings among states of the same strangeness but it does strongly break the SU_3^{fl} multiplets as expected.

Within the approximation (26) it becomes possible to replace equation (25) by the much simpler form

$$E_g^0 = (\alpha_c/R) O_g\, M\left((n_s/N)m_s R, (n_s/N)m_s R\right). \tag{27}$$

Mass calculations for the multiquark configurations Q^N are now particularly simple as every state involves a well-defined number of strange quarks, and equation (27) assumes the simple form

$$E_g^0 = (\alpha_c/R)\Delta\, M\big((n_s/N)m_s R, (n_s/N)m_s R\big)\,, \tag{28}$$

where Δ is as defined by equation (5).

The evaluation of the masses of the multiquark hadrons is made in terms of the four parameters B, α_c, Z_0 and m_s, which have been determined in the configurations Q^3 and $Q\bar{Q}$ as (DeGrand *et al.* 1975)

$$B^{\frac{1}{4}} = 0\cdot 146 \text{ GeV}, \qquad \alpha_c = 0\cdot 55, \qquad Z_0 = 1\cdot 84, \qquad m_s = 0\cdot 279 \text{ GeV},$$

with the non-strange quarks being taken as massless. The calculation of the masses then proceeds by evaluating equation (19) for an assumed bag radius R to give

$$E(R) = \tfrac{4}{3}\pi B R^3 + P/R\,, \tag{29}$$

where

$$P = -Z_0 + n_0\,\omega_0 + n_s\,\omega_s + \alpha_c\,\Delta\, M\big((n_s/N)m_s R, (n_s/N)m_s R\big)\,. \tag{30}$$

The boundary condition (18) may then be written as

$$R_0^4 + \frac{n_s m_s}{4\pi B}\left\{\frac{\mathrm{d}\omega_s}{\mathrm{d}(m_s R)} + \frac{\alpha_c \Delta}{N}\frac{\mathrm{d}}{\mathrm{d}(m_s R)} M\left(\frac{n_s}{N}m_s R, \frac{n_s}{N}m_s R\right)\right\}R_0 = \frac{P}{4\pi B}. \tag{31}$$

Explicit evaluation of the derivatives in equation (31) shows them to be remarkably constant with respect to $m_s R$ and, to a very good approximation,

$$\frac{\mathrm{d}\omega_s}{\mathrm{d}(m_s R)} = 0\cdot 69\,, \qquad \frac{\mathrm{d}}{\mathrm{d}(m_s R)} M\left(\frac{n_s}{N}m_s R, \frac{n_s}{N}m_s R\right) = -0\cdot 038\,. \tag{32}$$

Thus equation (31) may be replaced by

$$R_0^4 + \frac{n_s R_0}{4\pi B}\left(33\cdot 036 - \frac{\Delta}{N}\right)5\cdot 83\times 10^{-3} = \frac{P}{4\pi B}. \tag{33}$$

The quartic in R_0 may be solved to the first approximation in R_0 as

$$R_0^{(1)} = 3\cdot 6377\, P^{\frac{1}{4}} \tag{34a}$$

and hence

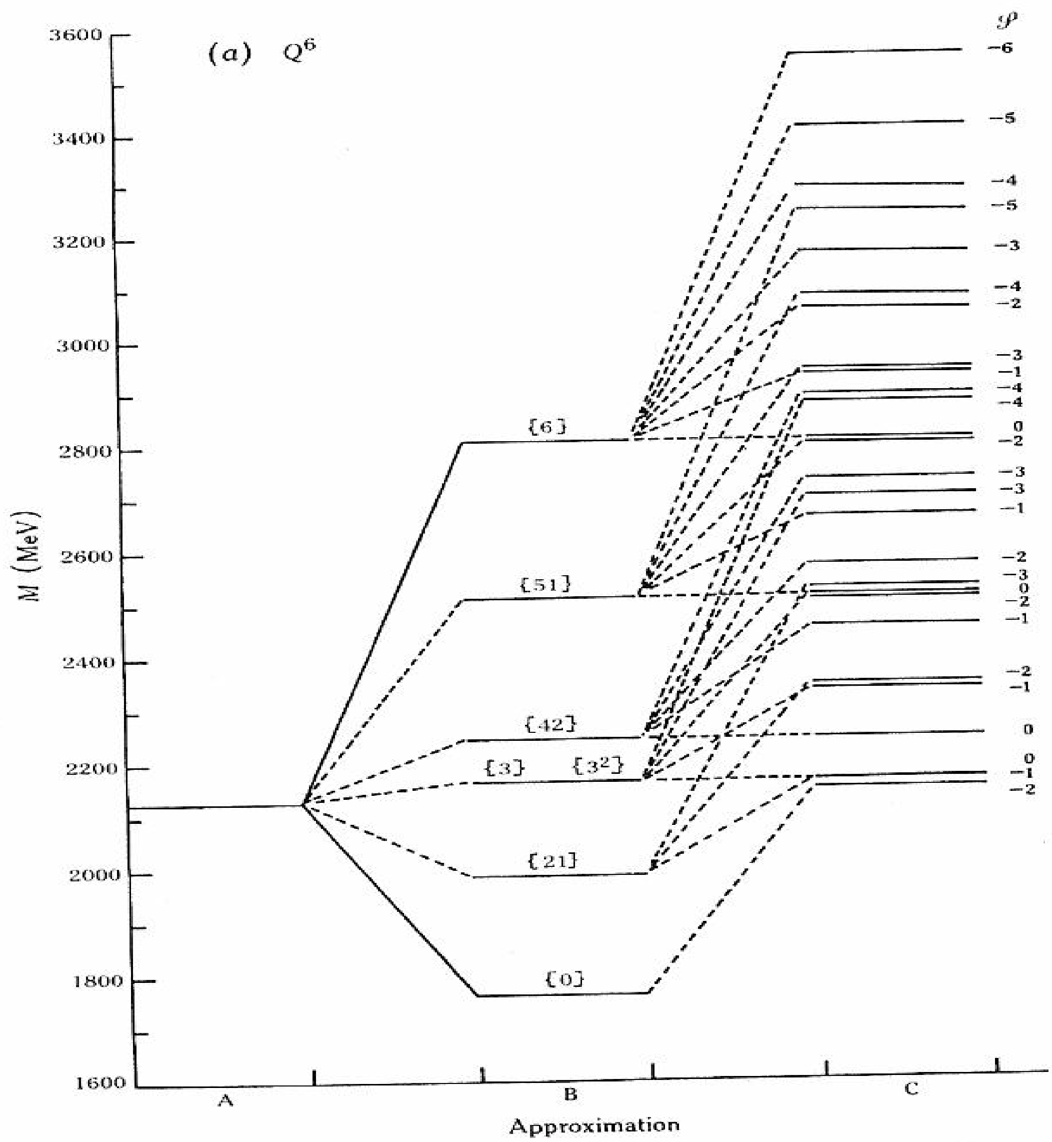
$$E_0^{(1)} = 366\cdot 52\, P^{\frac{3}{4}} \quad \text{MeV}\,. \tag{34b}$$

This value may be used then to give an improved value of R_0:

$$R_0^{(2)} = R_0^{(1)}\big(1 - 1\cdot 4575\, n_s\, R_0^{(1)}(33\cdot 036 - \Delta/N)10^{-3}P^{-1}\big)\,, \tag{35a}$$

$$E_0^{(2)} = \tfrac{4}{3}\pi B (R_0^{(2)})^3 + P/R_0^{(2)}\,. \tag{35b}$$

The procedure adopted to compute the masses of the Q^6 and Q^9 hadrons was to choose an initial value of R and evaluate P using equation (30). The resulting value of P was used in equations (34) and thence in (35) and the process continued until convergence was obtained.



Figs 1*a* and 1*b*. Calculated masses for the S-wave colour singlet states of the quark configurations (*a*) Q^6 and (*b*) Q^9. The strangeness number $\mathscr{S}$ associated with each state is indicated. The approximations represented are (see text):

A, no quark–gluon interaction;

B, quark–gluon interaction present, all quarks massless;

C, quark–gluon interaction present, strange quarks mass 279 MeV.

Application to Q^6 and Q^9

The results for the quark configurations Q^6 and Q^9 are displayed in Figs 1*a* and 1*b* respectively. In each case the first column A shows the mass obtained by neglecting the quark–gluon interaction. In this instance the states of the given configuration are completely degenerate. The second column B shows the effect of switching on the quark–gluon interaction assuming all the quarks are massless. In this approximation the SU_3^{fl} symmetry is exact. In the third column C the quark–gluon interaction is included with the strange quarks being given a mass of 279 MeV. This has the

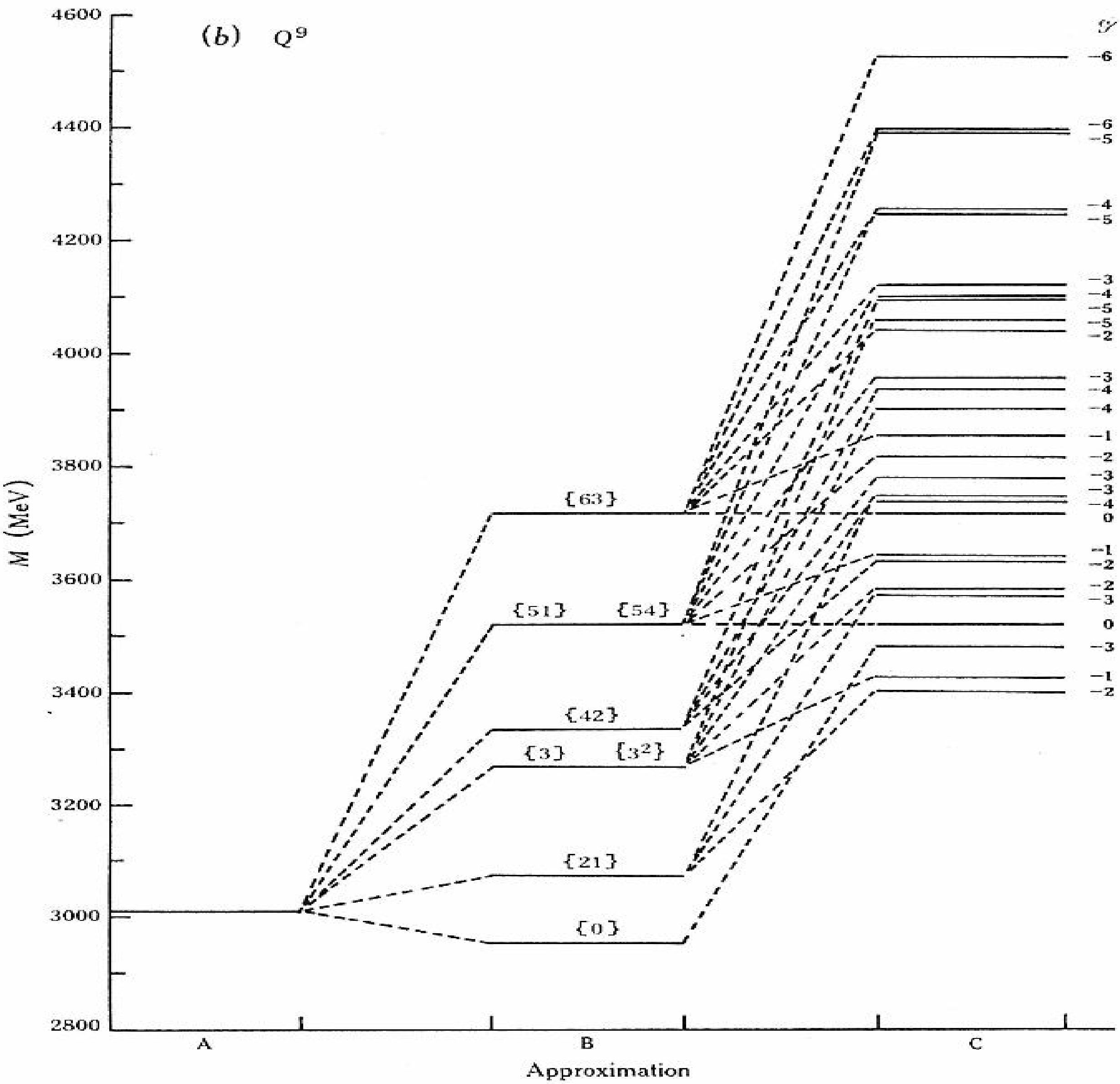


Fig. 1*b*

effect of strongly breaking the SU_3^{fl} symmetry down to $SU_2^I \times U_1^Y$ with states of the same strangeness being degenerate within a given SU_3^{fl} multiplet. The strangeness number $\mathscr{S}$ associated with each of these levels is indicated. It is emphasized that the calculations displayed in Figs 1*a* and 1*b* show only the states associated with the lowest spin S for a given SU_3^{fl} multiplet.

The magnetostatic quark–gluon interaction is primarily responsible for the mass splittings associated with a given quark configuration and may make positive or negative contributions to the masses of the colour singlet states. The states having negative contributions will tend to have greater stability especially if the resultant mass falls below the threshold for multibaryon decays. The quark–gluon interaction is necessarily positive for all the colour singlet S-wave states of the Q^N ($N = 0 \pmod 3$)

configurations with $N = 12$, 15 or 18 and thus no stable states are expected for $N > 9$. Any resonances or enhancements that did occur would necessarily be very weak and of great width.

The low mass states of the Q^6 and Q^9 configurations are of greatest interest as they have the possibility of exhibiting bound states and resonances. We consider first the states of Q^6.

The lowest mass state corresponds to a flavour singlet, with strangeness $\mathscr{S} = -2$ and $J^P = 0^+$, and involves the quark configuration *uuddss*. This state occurs approximately 85 MeV *below* twice the mass of the hyperon Λ and thus suggests the existence of a stable dihyperon (H) at ~ 2140 MeV. It is, however, important to realize that in the bag model this state is not simply a weakly bound state of two hyperons but rather a hadron in its own right corresponding to a six-quark colour singlet. It may be viewed as involving the combination of coloured and colourless three-quark states to produce the final colour singlet state. However, when the six-quark bag deforms and fissions, the decay products must involve only colourless objects. The possibility of a dihyperon resonance was suggested by Jaffe (1977*c*) but searches by Guy and Kadyk (1977) have failed to find it. Problems associated with the experimental production and observation of such resonances have recently been discussed by Lipkin (1977*a*, 1977*b*).

The lowest member of the flavour octet $\{21\}$ has strangeness $\mathscr{S} = -1$ with $J^P = 1^+$ and $I = \frac{1}{2}$. This level occurs ~ 100 MeV above the combined mass of pΛ. The experimental situation is by no means clear (Beilliere *et al.* 1976): enhancement in the Λp invariant mass has been observed at 2127 MeV but it is not clear whether it is due to a $B = 2$, $\mathscr{S} = -1$ resonance or a kinematic effect just below the Σn threshold. It is tempting to interpret the observed enhancement as being associated with the Q^6 configuration but this must await more conclusive experimental evidence.

The second member of the flavour octet has $\mathscr{S} = -2$ and $I = 1$ and 0 and could be expected to decay strongly into NΞ or $\Lambda\Sigma$. The flavour octet also has $J^P = 2^+$. These states all occur ~ 100 MeV above the corresponding $J^P = 1^+$ states and must involve dibaryons with one baryon of spin $\frac{1}{2}$ and one of spin $\frac{3}{2}$. No bound states are expected. Finally, we note that the lowest state of the $\{3^2\}$ flavour decuplet has the quantum numbers expected for the deuteron but with a mass ~ 275 MeV higher than for pn.

The prospects for observing bound states and resonances in Q^9 are much less encouraging. The lowest state is a member of a flavour octet $\{21\}$ with $\mathscr{S} = -2$, $I = \frac{1}{2}$ and $J^P = \frac{1}{2}^+$. This state occurs above any tribaryons having these quantum numbers and hence no bound states are expected. The octet occurs with $J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ and hence the density of levels is high and only weak resonances could arise if at all.

Concluding Remarks

We have described a simple method for determining the S-wave colour singlets for multiquark configurations and made a number of simplifications to the determination of their masses within the framework of the MIT bag model. No bound states have been found for the Q^N configurations where $N \geqslant 9$. There appears to be some possibility of observing one or two bound states or resonances in Q^6 associated with the SU_3^{f1} singlet $\{0\}$ and octet $\{21\}$. The successful observation of these states would add support to the QCD hypothesis. It would seem to be clear that there can be no bound states associated with any of the exotic SU_3^{fl} representations. While it is possible to refine these calculations further and compute electromagnetic mass differences etc., the results obtained so far suggest that in the absence of further experimental data this is likely to be an unrewarding task. The study of the quark–antiquark configurations $Q^x\bar{Q}^y$, especially $Q^4\bar{Q}$, is likely to be more rewarding as the masses are lower than those considered here and the magnetostatic gluon interaction is more favourable to the formation of bound states.

References

Beilliere, P., Wilquet, G., Guy, J. G., Tovey, S. N., and Knight, W. L. (1976). Univs Brussels (ULB–VUB) Rep. No. IIHE–76 . 9, November 1976.

Butler, P. H. (1975). *Philos. Trans. R. Soc. London* A **277**, 545.

Butler, P. H., and Wybourne, B. G. (1976). *Int. J. Quantum Chem.* **10**, 581, 599, 615.

Chodos, A., Jaffe, R. L., Johnson, K., and Thorn, C. B. (1974*b*). *Phys. Rev.* D **10**, 2599.

Chodos, A., Jaffe, R. L., Johnson, K., Thorn, C. B., and Weisskopf, V. F. (1974*a*). *Phys. Rev.* D **9**, 3471.

DeGrand, T., and Jaffe, R. L. (1976). *Ann. Phys. (New York)* **100**, 425.

DeGrand, T., Jaffe, R. L., Johnson, K., and Kiskis, J. (1975). *Phys. Rev.* D **12**, 2060.

Guy, J. G., and Kadyk, J. (1977). Rutherford Lab. Rep. No. RL–77–054/A.

Jaffe, R. L. (1977*a*). *Phys. Rev.* D **15**, 267.

Jaffe, R. L. (1977*b*). *Phys. Rev.* D **15**, 281.

Jaffe, R. L. (1977*c*). *Phys. Rev. Lett.* **38**, 195, 617.

Judd, B. R. (1963). 'Operator Techniques in Atomic Spectroscopy', p. 125 (McGraw–Hill: New York).

King, R. C. (1975). *J. Phys.* A **8**, 429.

Lipkin, H. J. (1977*a*). High Energy spectroscopy. Fermilab. Conf. No. 77/66–THY, July 1977.

Lipkin, H. J. (1977*b*). Theoretical review of strange and nonstrange mesons. Fermilab. Conf. No. 77/65–THY, July 1977.

Wybourne, B. G. (1970). 'Symmetry Principles and Atomic Spectroscopy' (with an Appendix of Tables by P. H. Butler) (Wiley–Interscience: New York).

Wybourne, B. G. (1974). 'Classical Groups for Physicists' (Wiley–Interscience: New York).

Wybourne, B. G., and Bowick, M. J. (1977). *Aust. J. Phys.* **30**, 259.

Manuscript received 26 September 1977