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S−functions applied to the theory of non-compact Lie groups

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And yet the mystery of mysteries is to view machines making machines; a spectacle that fills the mind with curious, and even awful, speculation.
— Benjamin Disraeli: *Coningsby* (1844)

1. Infinite Series of $S-$functions

$$A = \sum_\alpha (-1)^{\frac{a}{2}}\{\alpha\} \qquad B = \sum_\beta \{\beta\}$$
$$C = \sum_\gamma (-1)^{\frac{c}{2}}\{\gamma\} \qquad D = \sum_\delta \{\delta\}$$

$$E = \sum_\varepsilon (-1)^{(\frac{e+r}{2}}\{\varepsilon\} \qquad F = \sum_\zeta \{\zeta\}$$

$$L = \sum_m (-1)^m\{1^m\} \qquad M = \sum_m \{m\}$$

$$P = \sum_m (-1)^m\{m\} \qquad Q = \sum_m \{1^m\}$$

$$V = \sum_\omega (-1)^q\{\omega'\} \qquad W = \sum_\omega (-1)^q\{\omega\}$$

$$AB = CD = EF = LM = PQ = VW = 1 = \{0\}$$

$$PM = AD = W,\ LQ = BC = V,\ LA = PC = E$$

R C King, Luan Dehuai & B G Wybourne, J.Phys.A:Math.Gen 14, 2502 (1981)

2. Application to compact Lie groups

Unitary irreducible representations are all of finite dimension.

$$U(n) \to U(n-1)$$
$$\{\lambda\} \to \{\lambda/M\}$$
$$U(n) \to O(n)$$
$$\{\lambda\} \to [\lambda/D]$$
$$U(2n) \to Sp(2n)$$
$$\{\lambda\} \to \langle\lambda/B\rangle$$

$$Sp(2n) \to U(2n) \to O(n)$$
$$\langle\lambda\rangle \to \{\lambda/A\} \to [\lambda/AD]$$

$$Sp(2n) \to O(2n)$$
$$\langle\lambda\rangle \to [\lambda/W]$$
$$O(2n) \to Sp(2n)$$
$$[\lambda] \to \langle\lambda/V\rangle$$

G R E Black, R C King & B G Wybourne, J.Phys.A:Math.Gen 16, 1555 (1983)

3. The Non-compact groups $Sp(2n,\Re)$, $U(p,q)$ and $SO^*(2n)$

Non-trivial unitary irreducible representations are all of infinite dimension.

$$Sp(2n,\Re) \to U(n)$$
$$\langle \tfrac{1}{2}(0)\rangle \to \varepsilon^{\frac{1}{2}} M_+$$
$$\langle \tfrac{1}{2}(1)\rangle \to \varepsilon^{\frac{1}{2}} M_-$$

$$M_+ = \sum_{m=0}^{\infty} \{2m\}$$
$$M_- = \sum_{m=0}^{\infty} \{2m+1\}$$

Physically $Sp(6,\Re)$ is the dynamical group of the three-dimensional harmonic oscillator.

R C King & B G Wybourne, J.Phys.A:Math.Gen 18 3113 (1985)

4. The discrete harmonic series irreducible representations

The groups $Sp(2n,\Re)$ and $O(k)$ form a dual pair with respect to the metaplectic group $Mp(2nk,\Re)$ such that the basic irreducible representation $\tilde{\Delta}$ under $Sp(2nk,\Re) \to Sp(2n,\Re) \times O(k)$ branches as

$$\tilde{\Delta} \to \sum_{\lambda} \langle \tfrac{1}{2}k(\lambda)\rangle \times [\lambda]$$

where the summation is over all λ such that

$$\lambda_1' + \lambda_2' \leq k \quad \textbf{and} \quad \lambda_1' \leq n$$

Likewise, for the dual pair $SO^*(2n),\ Sp(2k)$ we have

$$\tilde{\Delta} \to \sum_{\lambda} [k(\lambda)] \times \langle\lambda\rangle$$

where the summation is over all λ such that

$$\lambda_1' \leq \min(n,k)$$

5. Branching Rules for non-compact groups

$$Sp(2n,\Re) \to U(n)$$
$$\langle \tfrac{1}{2}k(\lambda)\rangle \to \varepsilon^{\frac{k}{2}} \cdot \{\{\lambda_s\}_N^k \cdot D_N\}_N$$

$N = \min(n,k)$

$$SO^*(2n) \to U(n)$$
$$[k(\lambda)] \to \varepsilon^k \cdot \{\{\lambda_s\}_N^{\langle 2k\rangle} \cdot B_N\}_N$$

where $N = min(2k,n)$**.**

**R C King & B G Wybourne,
J.Phys.A:Math.Gen 31,6691 (1998)**

**R C King, F. Toumazet & B G Wybourne,
J.Phys.A:Math.Gen 31, 6691 (1998)**

6. Physical applications require the analysis of plethysms of irreducible representations of non-compact groups.

We find for the second powers of the basic irreducible representations of $Sp(2n,\Re)$:-

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{2\} = \sum_{i=0}^{\infty} \langle 1(4i)\rangle \tag{1}$$

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} = \sum_{i=0}^{\infty} \langle 1(2+4i)\rangle \tag{2}$$

$$\langle \tfrac{1}{2}(1)\rangle \otimes \{2\} = \sum_{i=0}^{\infty} \langle 1(2+4i)\rangle \tag{3}$$

$$\langle \tfrac{1}{2}(1)\rangle \otimes \{1^2\} = \langle 1(1^2)\rangle + \sum_{i=1}^{\infty} \langle 1(4i)\rangle \tag{4}$$

Eq. (2) and (3) imply

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} \equiv \langle \tfrac{1}{2}(1)\rangle \otimes \{2\} \tag{5}$$

which implies the $S-$function identity

$$M_+ \otimes \{1^2\} \equiv M_- \otimes \{2\} \tag{6}$$

7. Further $S-$function plethysm identities can be found.

Define

$$A_\pm = \{1^2\} \otimes L_\pm \quad B_\pm = \{1^2\} \otimes M_\pm$$
$$C_\pm = \{2\} \otimes L_\pm \quad D_\pm = \{2\} \otimes M_\pm$$

Let $Z = A, B, C, D, M, L$ **then**

$$Z_+ \otimes \{1^2\} = Z_- \otimes \{2\}$$

and

$$Z \otimes \{2\} = ZZ_+ \quad \textbf{and} \quad Z \otimes \{1^2\} = ZZ_-$$

Furthermore

$$Z_+ \otimes \{21^2\} = Z_- \otimes \{31\}$$

to which can be added the generalisation

$$Z_+ \otimes (\{1^2\} \otimes \{\sigma\} = Z_- \otimes (\{2\} \otimes \{\sigma\}$$

K Grudzinski & B G Wybourne
J.Phys.A:Math.Gen 29, 6631 (1996)

J-Y Thibon, F Toumazet & B G Wybourne
J.Phys.A:Math.Gen 31, 1073 (1998)

8. Patterns in plethysms for $Sp(2n,\Re)$

$\langle\frac{1}{2}(0)\rangle \otimes \{3\} =$

$$< s1;(0) > \quad + < s1;(4) > \quad + < s1;(6) >$$
$$+ < s1;(8) > \quad + < s1;(91) > \quad \ldots$$

$\langle\frac{1}{2}(1)\rangle \otimes \{1^3\} =$

$$< s1;(1^3) > \quad + < s1;(41) > \quad + < s1;(61) >$$
$$+ < s1;(81) > \quad + < s1;(9) > \quad \ldots$$

$\langle 1(0)\rangle \otimes \{3\} =$

$$< 3;(0) > \quad + < 3;(2^2) > \quad + < 3;(3^2 1^2) >$$
$$+ < 3;(4) > \quad + < 3;(42) > \quad + < 3;(42^2) >$$
$$+ < 3;(521) > \quad \ldots$$

$\langle 1(1^2)\rangle \otimes \{3\} =$

$$< 3;(1^6) > \quad + < 3;(2^2 1^2) > + < 3;(3^2) >$$
$$+ < 3;(41^4) > \quad + < 3;(421^2) > + < 3;(42^2) >$$
$$+ < 3;(521) > \quad \ldots$$

Theorem:- For any partition $\rho \vdash r$, the corresponding $r-$fold symmetrized power of the associate irreducible representation $\langle\frac{1}{2}k(\lambda)\rangle^*$ of $Sp(2n,\Re)$ is such that

$$\langle\tfrac{1}{2}k(\lambda)\rangle^* \otimes \{\rho\} = \begin{cases} \left(\langle\tfrac{1}{2}k(\lambda)\rangle \otimes \{\rho\}\right)^* & \textbf{if } k \textbf{ is even;} \\ \left(\langle\tfrac{1}{2}k(\lambda)\rangle \otimes \{\rho'\}\right)^* & \textbf{if } k \textbf{ is odd,} \end{cases} \tag{5.25}$$

where the * on the left signifies a k-associate, while those on the right signify kr-associates.

R C King & B G Wybourne,
J.Phys.A:Math.Gen 31,6691 (1998)

9. Relations between group chains and irreducible representations of $SO^*(2n)$ and $Sp(2n\Re)$

$$
\begin{array}{ccccc}
SO^*(2n)\times Sp(2k) & \longleftarrow & Mp(4nk) & \longrightarrow & Sp(2n,\Re)\times O(2k) \\
\Big\downarrow & & & & \Big\downarrow \\
U(n)\times Sp(2k) & & & & U(n)\times O(2k) \\
\Big\downarrow & & & & \Big\downarrow \\
U(n)\times SO(2k) & \longrightarrow & & \longleftarrow & U(n)\times SO(2k)
\end{array}
$$

R C King, F. Toumazet & B G Wybourne, J.Phys.A:Math.Gen 31, 6691 (1998)

$$
\begin{array}{ccccc}
U(p,q)\times U(k) & \longleftarrow & Mp(2(p+q)k) & \longrightarrow & Sp(2n,\Re)\times O(k) \\
\downarrow & & & & \downarrow \\
U(p,q)\times O(k) & & & & U(p+q)\times O(k) \\
\downarrow & & & & \downarrow \\
U(p)\times U(q)\times O(k) & \longrightarrow & & \longleftarrow & U(p)\times U(q)\times O(k)
\end{array}
$$

10. Plethysm identities for $U(p,q)$

The group $U(p,q)$ contains an infinite set of basic irreducible representations

$$H = \{1(\bar{0};0)\} + \sum_{m=1}^{\infty} (\{1(\bar{m};0)\} + \{1(\bar{0};m)\})$$

which may be conveniently divided as

$$H = H^+ + H^-$$

where

$$H^+ = H_0 + \sum_{m=1}^{\infty} (H_{2m} + H_{-2m})$$

$$H^- = \sum_{m=1}^{\infty} (H_{2m+1} + H_{-2m-1})$$

with $H_x = \{1(\bar{0};x)\}$ **and** $H_{-x} = \{1(\bar{x};0)\}$

It is not difficult to see from the earlier diagram that in terms of their $U(p)\times U(q)$ decompositions that

$$\langle \tfrac{1}{2}(0)\rangle \sim H^{+} \quad \textbf{and} \quad \langle \tfrac{1}{2}(1)\rangle \sim H^{-}$$

Recalling that

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} \equiv \langle \tfrac{1}{2}(1)\rangle \otimes \{2\}$$

leading immediately to the non-trivial plethysm identity for $U(p,q)$

$$H^{+} \otimes \{1^2\} \equiv H^{-} \otimes \{2\}$$

11. Combinatorial identities

The properties of Lie groups can provide a rich supply of combinatorial identities of which I give one for illustrative purposes. It was proved by the late Derek Breach and the proof lost, given a machine proof, and again rederived by R C King.

$$3^{\nu-1} = \sum_x \left\{ \binom{2\nu}{\nu-1-6x} - \binom{2\nu}{\nu-2-6x} - \binom{2\nu}{\nu-4-6x} + \binom{2\nu}{\nu-5-6x} \right\}$$

R C King, Luan Dehuai & B G Wybourne, J.Phys.A:Math.Gen 14, 2502 (1981)

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which returns us to the beginning to Benjamin Disraeli's "Coningsby"

Collaborators

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Questions?

The only questions worth asking are the unanswerable ones
— John Ciardi *Saturday Review-World (1973)*

For every complex question there is a simple answer — and it's wrong.
— H. L. Mencken