The Heritage of Sophus Lie

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The contributions of a scientist are not to be seen purely in terms of the work done during a lifetime but in their continuing influence and in new work that has grown out from them. It is the latter features of Lie's work I wish to note today. An individuals work necessarily draws on previous work of others. Lie is no exception. There was the calculus developed, furiously, and with considerable rancour, by Newton and Liebnitz. Newton in his theory of equations introduced the concept of symmetric functions and transformations between different types of such functions. Galois, in his all too short and stupidly ended life, developed the concept of the group.

Lie's collected works fill some three volumes with much devotion to the introduction of the concept of the infinitesimal transformation about the identity. These played a vital role in Lie's interest in invariant transformations and those transformations that left equations form invariant. These allowed the possibilty of transforming solutions of differential equations into new solutions. The discussion of the commutators of infinitesimal operators that generated infinitesimal transformations led to the *Lie algebras* while the generation of finite transformations formed the global groups, the *Lie groups*.

Killing and Cartan completed the classification of the complex semisimple Lie algebras with Cartan, in his thesis of 1894, giving the now familiar letter designations for the Lie algebras and Lie groups as:-

Algebra	A_{n-1}	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
Group	SU_n	SO_{2n+1}	Sp_{2n}	SO_{2n}	G_2	F_4	E_6	E_7	E_8

Cartan started the representation theory of Lie algebras with his introduction of the concept of a *highest weight* which was to come to full fruition at the hands of Hermann Weyl.

Almost as an interlude Emmy Nöether saw the relationship between the invariance transformations of Lie and the conservation laws of the physicists. Realising that every conservation law amounts to the statement of an impossible experiment.

Frobenius and Schur continued the development of the theory of symmetric functions and their relationship to the representation and character theory of the symmetric group as well as to those of the general linear group. Alfred Young, virtually working alone, completed the representation theory of the ordinary representations of the symmetric group. It was Weyl who first fully appreciated Young's work and saw its significance for Lie groups.

H. Casimir and B. van der Waerden start the investigation of the subgroups and subalgebras of the Lie algebras and Lie groups with Casimir returning to Lie's ideas of invariance to develop what is now known as the Casimir operator. At about the same time D. E. Littlewood and F. D. Murnaghan were applying the earlier work of Schur to calculating properties of Lie group representations and characters.

E. Wigner developed the representation theory of the little Lorentz group and thus non-compact Lie groups made their entry into physics. This was later to be taken up by mathematicians such as Harish Chandra who developed the concept of tempered characters for the infinite dimensional unitary representations of non-compact groups which now find many applications in physics in diverse fields such as nuclear theory, quantum optics etc.

G. Racah's celebrated Princeton lectures did much to bring the subject of Lie groups to the attention of physicists. His work, while directed at the atomic physics of the f-shell of the rare earth elements was most rapidly seized upon by the nuclear theorists and later by the particle physicists. The development of the Lie group SU_3 first in nuclear theory and later in particle physics is very much part of the Sophus Lie heritage.

The introduction of second-quantisation was eventually to lead to physicists constructing Lie algebras using commutators for bosons and anticommutators for fermions. Only in recent times did it occur to physicists to combine both to form Lie supergroups. The formation of these new structures has led to further attempts to more general structures in which Lie's original groups and algebras appear as special cases and is perhaps the ultimate accolade to be paid to Lie. q-deformations and so-called quantum groups, a feature of this workshop, bear testimony to the heritage of Sophus Lie. 150 years since his birth his work continues to exercise a profound influence on modern developments in chemistry, physics and mathematics.