Products and symmetrised powers of irreducible representations of $SO^*(2n)$

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Abstract. The calculation of branching rules, tensor products and plethysms of the infinite dimensional harmonic series unitary irreducible representations of the non-compact group $SO^*(2n)$ is considered and the duality between $SO^*(2n)$ and $Sp(2k)$ exploited. The branching rule for the restriction of an arbitrary harmonic series irreducible representation of $SO^*(2n)$ to $U(n)$ is derived, and the decomposition is given explicitly for each of the infinite number of fundamental harmonic series irreducible representations, H_m, of $SO^*(2n)$ whose direct sum constitutes the metaplectic representation, H, of $SO^*(2n)$. A concise expression for the decomposition of tensor products is derived and a complete analysis of the terms in both $H_m \times H_{m'}$ and $H \times H$ is given. A general formula for plethysms of arbitrary irreducible representations of $SO^*(2n)$ is derived and its implementation illustrated both by means of a detailed generic example and by a complete determination of the symmetric and antisymmetric terms of $H \times H$. Finally, relationships that arise from the embedding of the product groups $SO^*(2n) \times Sp(2k)$ and $Sp(2n, \Re) \times O(2k)$ in the metaplectic group $Mp(4nk)$ are discussed.

1. Introduction

The group $SO^*(2n)$ occurs as a maximal non-compact subgroup of the metaplectic group, $Mp(2n)$, which is also the double covering group for the non-compact symplectic group $Sp(2n, \Re)$ which finds important applications as the dynamical group of the harmonic oscillator. The group $SO^*(8)$ has been considered in the literature[1,2] with the local isomorphism $SO(6,2) \sim SO^*(8)$ being exploited to show the existence of a complete set of $SU(3)$ tensor operators in the enveloping algebra of $SO^*(8)$. Here we wish to discuss the general case of the groups $SO^*(2n)$ which leave the skew Hermitean form

$$- z_1 z_{n+1}^* + z_{n+1} z_1^* - \ldots - z_n z_{2n}^* + z_{2n} z_n^*$$

invariant[3]. A preliminary account of the group $SO^*(2n)$ was given earlier[4]. That paper was largely devoted to the holomorphic discrete series and harmonic series of

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the non-compact groups $Sp(2n,\Re)$ and $U(p,q)$ with detailed derivations of appropriate branching rules and tensor products. Only scant attention was paid to $SO^*(2n)$ and the authors concluded their paper with the remark: "Little attention has been given to $SO^*(2n)$ but we suspect that comparable formulae can be derived in this case, it being merely necessary to change D to B and modify other rules appropriately". In this paper we explore in some detail the properties of the irreducible representations of $SO^*(2n)$ and obtain the non-trivial "merely necessary" changes. Recent work[5−7] has shed further light on the properties of $Sp(2n,\Re)$ and $U(p,q)$, and more recently still[8] it has been found convenient to introduce the notion of associate irreducible representations of $Sp(2n,\Re)$. While such a notion owes its origin to the existence of mutually associate pairs[11,12] of irreducible representations of $SO(2k)$, such irreducible representations do not exist for $SO^*(2n)$. Nonetheless, we show herein that one can map self-associate finite sets of irreducible representations of $Sp(2n,\Re)$ into infinite sets of irreducible representations of $SO^*(2n)$.

Formulae are given for the evaluation of branching rules, tensor products and plethysms for arbitrary unitary harmonic series irreducible representations of $SO^*(2n)$. These irreducible representations all appear as constituents of some power of the harmonic representation, H, of $SO^*(2n)$. The representation H is the restriction to $SO^*(2n)$ of the irreducible metaplectic representation, $\tilde{\Delta}$, of the metaplectic group $Mp(2n)$. H is itself reducible into a direct sum of an infinite number of fundamental irreducible representations in accordance with the decomposition:

$$H = \sum_{m=0}^{\infty} H_m = \sum_{m=0}^{\infty} [1(m)], \tag{1.1}$$

where it has been convenient to denote each of the infinite-dimensional fundamental irreducible representations H_m of $SO^*(2n)$ by $[1(m)]$. It is also convenient to write $H = H_+ + H_-$ with

$$H_+ = \sum_{k=0}^{\infty} [1(2k)] \quad \text{and} \quad H_- = \sum_{k=0}^{\infty} [1(2k+1)]. \tag{1.2}$$

Relatively simple expressions are obtained for the branching rules, tensor products and plethysms involving H and its various constituents $H_\pm$ and H_m for $m = 0, 1, \ldots, \infty$. In particular, complete results are given for the terms in $H^2 = H \times H$ and for those in the symmetric and antisymmetric parts, $H \otimes \{2\}$ and $H \otimes \{1^2\}$, respectively, of H^2.

Throughout we follow the notation developed earlier[4,5,8,9] for representations of non-compact groups and certain signed sequences. In the case of the notation for partitions and symmetric functions we follow that of Macdonald[10], while for S-functions series and modification rules we call where appropriate on formulae collected together in three previous articles[11−13]. References [4] and [8] will often be designated as KW1 and KW2, respectively.

2. $SO^*(2n) \to U(n)$ branching rule

It should first be noted that the non-compact group $SO^*(2n)$ and the compact group $Sp(2k)$ are a dual pair with respect to $Mp(4nk)$ in the sense that each is a maximal centraliser of the other in the metaplectic group $Mp(4nk)$. As a direct consequence of this, the metaplectic representation of $Sp(4nk,\Re)$ decomposes under restriction to $SO^*(2n)\times Sp(2k)$ in accordance with the rule:

$$\tilde{\Delta}\to\sum_{\lambda}\,[k(\lambda)]\times\langle\lambda\rangle, \tag{2.1}$$

where the summation is over all λ such that

$$\lambda_1'\le k \quad\text{and}\quad \lambda_1'\le n. \tag{2.2}$$

The procedure necessary to determine the $U(n)$ content of each harmonic series irreducible representation $[k(\lambda)]$ of $SO^*(2n)$ is closely related to that used for $Sp(2n,\Re)$ in Section 5 of KW1. Consider the two group-subgroup chains:

$$Sp(4nk,\Re)\to SO^*(2n)\times Sp(2k)\to U(n)\times Sp(2k); \tag{2.3a}$$
$$Sp(4nk,\Re)\to U(2nk)\to U(n)\times U(2k)\to U(n)\times Sp(2k). \tag{2.3b}$$

The first of these gives

$$\tilde{\Delta}\to\sum_{\lambda}\,[k(\lambda)]\times\langle\lambda\rangle\to\sum_{\lambda,\mu}\varepsilon^k B^{\mu}_{\lambda}\{\mu\}\times\langle\lambda\rangle, \tag{2.4}$$

while the second leads to

$$\tilde{\Delta}\to\sum_{m}\varepsilon^{1/2}\{m\}\to\sum_{\mu}\varepsilon^k\{\mu\}\times\varepsilon^{n/2}\{\mu\}\to\sum_{\mu,\lambda}\varepsilon^k R^{\mu}_{\lambda}\{\mu\}\times\langle\lambda\rangle, \tag{2.5}$$

or, equivalently,

$$\begin{aligned}\tilde{\Delta}\to\sum_{m}\varepsilon^{1/2}\{m\}\to\sum_{\mu}\varepsilon^k\{\mu\}\times\varepsilon^{n/2}\{\mu\}&\to\sum_{\mu}\varepsilon^k\{\mu\}\times\langle\mu/B\rangle\\ &=\sum_{\nu}\varepsilon^k\{\nu\cdot B\}\times\langle\nu\rangle,\end{aligned} \tag{2.6}$$

where[11−13]

$$B=\sum_{\beta}\{\beta\}=\{0\}+\{1^2\}+\{2^2\}+\{1^4\}+\{3^2\}+\{2^21^2\}+\{1^6\}+\cdots, \tag{2.7}$$

with the summation taken over all partitions β such that each distinct part is repeated an even number of times. In (2.4) the coefficients B^{μ}_{λ} are the required branching rule coefficients for $SO^*(2n)\to U(n)$, while in (2.5) the coefficients R^{μ}_{λ} are the known branching rule coefficients for $U(2k)\to Sp(2k)$. These are defined implicitly by (2.6). We thus arrive at the following:

Proposition 2.1 *Let λ be such that $\lambda_1'\le\min(k,n)$. Then on restriction from $SO^*(2n)$ to $U(n)$ the irreducible representation $[k(\lambda)]$ of $SO^*(2n)$ decomposes in accordance with the branching rule:*

$$[k(\lambda)]\to\sum_{\mu}\varepsilon^k R^{\mu}_{\lambda}\{\mu\}=\varepsilon^k\cdot\{\lambda_s\}^{\langle 2k\rangle}\cdot B, \tag{2.8}$$

where $\{\lambda_s\}^{\langle 2k\rangle}$ *is the signed sequence*[4,5,8,9]

$$\{\lambda_s\}^{\langle 2k\rangle} = \sum_{\nu} \xi_\nu^\lambda \{\nu\} \tag{2.9}$$

with the summation extending over all ν *with* $\nu_1' \leq 2k$ *such* $\langle\nu\rangle = \xi_\mu^\lambda \langle\lambda\rangle$ *under the modification rules of* $Sp(2k)$*.*

The superscript $\langle 2k\rangle$ has been used as a notational device to emphasise that the signed sequences are constructed from a knowledge of the modification rules of $Sp(2k)$. These rules[11,13] are such that the non-vanishing coefficients ξ_ν^λ are all ± 1.

It follows from further consideration of the limitations imposed by branching via $U(n) \times U(2k)$ in (2.5) and (2.6) that (2.8) can be re-written in the computational simpler form:

$$[k(\lambda)] \rightarrow \varepsilon^k \cdot \{\{\lambda_s\}_N^{\langle 2k\rangle} \cdot B_N\}_N. \tag{2.10}$$

where $N = min(2k, n)$. The first $\cdot$ indicates a product in $U(n)$ and the second $\cdot$ a product in $U(N)$ as implied by the various subscripts N which limit all terms to those labelled by partitions into no more than N parts.

Irreducible representations $[k(\lambda)]$ of $SO^*(2n)$ satisfying (2.2) will be said to be *standard*. The signed sequence $\{\lambda_s\}_N^{\langle 2k\rangle}$ associated with modifications in $Sp(2k)$ is rendered finite by the constraint implied by the subscript N. Thus for $k = 2$ and $n = 4$ we have $N = 4$ and, for example,

$$\{31\}_4^{\langle 4\rangle} = \{31\} - \{31^3\}, \tag{2.11a}$$

whereas for $k = 2$ and $n = 3$ we have $N = 3$ and only the first term survives:

$$\{31\}_3^{\langle 4\rangle} = \{31\}. \tag{2.11b}$$

In general the modification rules[11,12] for $Sp(2k)$ are such that for each standard irreducible representation $[k(\lambda)]$ of $SO^*(2n)$ the corresponding signed sequence takes the form:

$$\{\lambda_s\}^{\langle 2k\rangle} = \{\lambda\} - \{\nu\} \pm \{\rho\} + \cdots \quad \text{with} \quad \lambda_1' < \nu_1' \leq \rho_1' \leq \cdots. \tag{2.12a}$$

where

$$\{\nu'\} = \{2k + 2 - \lambda_1', \lambda_2', \lambda_3', \ldots\}. \tag{2.12b}$$

Standard irreducible representations $[k(\lambda)]$ associated with signed sequences $\{\lambda_s\}_N^{2k}$ involving just one term are said to be *highly standard*. From (2.12) it can be seen that this will be the case whenever

$$2k + 2 - \lambda_1' > N = \min(n, 2k). \tag{2.13}$$

In particular this condition is automatically satisfied if $\lambda_1' \leq 1$. Hence all the irreducible representations $[k(m)]$ are highly standard, including $H_m = [1(m)]$ for all m.

More generally, for all highly standard irreducible representations of $SO^*(2n)$ (2.10) simplifies to just

$$[k(\lambda)] \to \varepsilon^k \cdot \{\{\lambda\} \cdot B_N\}_N \tag{2.14}$$

For example, the highly standard irreducible representation $[2(31)]$ of $SO^*(6)$ branches under $SO^*(6) \to U(3)$ as

$$\begin{aligned}[2(31)] &\to \varepsilon^2 \cdot \{\{31\} \cdot B_3\}_3 \\ &= \varepsilon^2 \cdot \{\{31\} \cdot (\{0\} + \{1^2\} + \{2^2\} + \{3^2\} + \{4^2\} + \cdots)\}_3 \\ &= \varepsilon^2 \cdot (\{31\} + \{321\} + \{3^2 2\} + \{41^2\} + \{42\} + \{42^2\} \\ &\quad + \{431\} + \{521\} + \{53\} + \cdots) \\ &= \{532\} + \{543\} + \{5^2 4\} + \{63^2\} + \{642\} + \{64^2\} \\ &\quad + \{653\} + \{743\} + \{752\} + \cdots . \end{aligned} \tag{2.15}$$

In the case of the standard, but not highly-standard irreducible representation $[2(31)]$ of $SO^*(8)$ we have from (2.10) and (2.11), for $SO^*(8) \to U(4)$

$$\begin{aligned}[2(31)] &\to \varepsilon^2 \cdot \{(\{31\} - \{31^3\}) \cdot B_4\}_4 \\ &= \varepsilon^2 \cdot \{(\{31\} - \{31^3\}) \cdot (\{0\} + \{1^2\} + \{1^4\} + \{2^2\} + \{2^2 1^2\} \\ &\quad + \{2^4\} + \{3^2\} + \{3^2 1^2\} + \{4^2\} + \cdots)\}_4 \\ &= \varepsilon^2 \cdot (\{31\} + \{321\} + \{3^2 2\} + \{41^2\} + \{42\} + \{421^2\} \\ &\quad + \{42^2\} + \{431\} + \{521\} + \{53\} + \cdots) \\ &= \{532^2\} + \{5432\} + \{5^2 42\} + \{63^2 2\} + \{642^2\} + \{643^2\} \\ &\quad + \{64^2 2\} + \{6532\} + \{7432\} + \{752^2\} + \cdots . \end{aligned} \tag{2.16}$$

In the particular case of the fundamental irreducible representations H_m of $SO^*(2n)$ we have

$$H_m = [1(m)] \to \varepsilon \cdot \{\{m\} \cdot B_2\}_2 = \sum_{r=0}^{\infty} \varepsilon \cdot \{m{+}r, r\} = \sum_{r=1}^{\infty} \{m{+}r, r\}, \tag{2.17}$$

since $B_2 = \sum_{r=0}^{\infty} \{r^2\}$ and the relevant products are taken in $U(2)$ with $\varepsilon = \{1^2\}$. It follows that under $SO^*(2n) \to U(n)$ the basic harmonic representation decomposes as

$$H = \sum_{m=0}^{\infty} [1(m)] \to \sum_{m,r=0}^{\infty} \varepsilon \cdot \{m{+}r, r\}. \tag{2.18}$$

Equivalently, but more formally, we have

$$H = \sum_{m=0}^{\infty} [1(m)] \to \sum_{m=0}^{\infty} \varepsilon \cdot \{\{m\} \cdot B_2\}_2 = \varepsilon \cdot \{M B_2\}_2 = \varepsilon \cdot F_2, \tag{2.19}$$

where quite generally[11−13]

$$M = \sum_m \{m\} \quad \text{and} \quad MB = F = \sum_\zeta \{\zeta\}, \tag{2.20}$$

with the latter sum being over taken over all partitions ζ, although in (2.19) the subscript on F_2 indicates that the series is to be restricted to partitions involving at most two parts.

3. Tensor products for harmonic unirreps of $SO^*(2n)$

The case of tensor products for the holomorphic discrete series of $SO^*(2n)$ was considered in KW1 who gave the result as (KW1 (7.12))

$$[\{\mu\}] \times [\{\nu\}] = [\{\mu \cdot \nu \cdot B\}] \tag{3.1}$$

The corresponding results for the harmonic irreducible representations follow in a very similar fashion to KW1 (8.10 - 8.15) by consideration of the two group-subgroup chains

$$\begin{aligned} Sp(4nk{+}4n\ell, \Re) &\to Sp(4nk, \Re) \times Sp(4n\ell, \Re) \\ &\to SO^*(2n) \times Sp(2k) \times SO^*(2n) \times Sp(2\ell) \\ &\to SO^*(2n) \times Sp(2k) \times Sp(2\ell) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} Sp(4nk{+}4n\ell, \Re) &\to SO^*(2n) \times Sp(2k+2\ell) \\ &\to SO^*(2n) \times Sp(2k) \times Sp(2\ell). \end{aligned} \tag{3.3}$$

Under (3.2) we have

$$\begin{aligned} \tilde{\Delta} \to \tilde{\Delta} \times \tilde{\Delta} &\to \sum_{\mu,\nu} [k(\mu)] \times \langle\mu\rangle \times [\ell(\nu)] \times \langle\nu\rangle \\ &\to \sum_{\mu,\nu,\lambda} K^{\mu\nu}_{\lambda} [k{+}\ell(\lambda)] \times \langle\mu\rangle \times \langle\nu\rangle, \end{aligned} \tag{3.4}$$

where $K^{\mu\nu}_{\lambda}$ are the required tensor product coefficients for $SO^*(2n)$. Alternatively, under (3.3) we have

$$\begin{aligned} \tilde{\Delta} &\to \sum_{\lambda} [k{+}\ell(\lambda)] \times \langle\lambda\rangle \\ &\to \sum_{\lambda} [k{+}\ell(\lambda)] \times \sum_{\mu,\nu} R^{\mu\nu}_{\lambda} \langle\mu\rangle \times \langle\nu\rangle, \end{aligned} \tag{3.5}$$

where the coefficients $R^{\mu\nu}_{\lambda}$ are the branching rule coefficients for the restriction $Sp(2k{+}2\ell) \to Sp(2k) \times Sp(2\ell)$. Comparison of (3.4) and (3.5) shows that $K^{\mu\nu}_{\lambda} = R^{\mu\nu}_{\lambda}$, thereby yielding:

Proposition 3.1 *The tensor product of a pair of unitary harmonic irreducible representations $[k(\mu)]$ and $[\ell(\nu)]$ of $S)^*(2n)$ decomposes in accordance with the rule*

$$[k(\mu)] \times [\ell(\nu)] = \sum_{\lambda} R^{\mu\nu}_{\lambda} \; [k{+}\ell(\lambda)]. \tag{3.6}$$

To implement (3.6) it is convenient to note that under the restriction $Sp(2k{+}2\ell) \to Sp(2k) \times Sp(2\ell)$ we have

$$\langle\lambda\rangle \to \sum_{\mu,\nu} R^{\mu\nu}_{\lambda} \; \langle\mu\rangle \times \langle\nu\rangle = \sum_{\kappa} \langle\lambda/\kappa\rangle \times \langle\kappa/B\rangle. \tag{3.7}$$

This may be derived through the use of standard S-function method[11].

An alternative formula may be derived from a consideration of the group-subgroup chains:

$$SO^*(2n) \times SO^*(2n) \to U(n) \times U(n) \to U(n); \tag{3.8a}$$

$$SO^*(2n) \times SO^*(2n) \to SO^*(2n) \to U(n). \tag{3.8b}$$

Using (2.8) the first of these gives

$$\begin{aligned}[k(\mu)] \times [\ell(\nu)] &\to \left(\varepsilon^k \cdot \{\mu_s\}^{\langle 2k\rangle} \cdot B\right) \times \left(\varepsilon^\ell \cdot \{\nu_s\}^{\langle 2\ell\rangle} \cdot B\right) \\ &\to \varepsilon^{k+\ell} \cdot \left(\{\mu_s\}^{\langle 2k\rangle} \cdot \{\nu_s\}^{\langle 2\ell\rangle} \cdot B\right) \cdot B,\end{aligned} \tag{3.9}$$

while from (3.6) and the use once more of (2.8) we obtain

$$\begin{aligned}[k(\mu] \times [\ell(\nu)] &= \sum_\lambda R^{\mu\nu}_\lambda [k{+}\ell(\lambda)] \\ &\to \sum_\lambda R^{\mu\nu}_\lambda \varepsilon^{k+\ell} \cdot \{\lambda_s\}^{\langle 2(k+\ell)\rangle} \cdot B,\end{aligned} \tag{3.10}$$

with $\lambda'_1 \leq N = \min(k{+}\ell, n)$. Comparison of (3.9) and (3.10) then yields:

Proposition 3.2 *The tensor product of a pair of unitary harmonic irreducible representations of* $SO^*(2n)$ *decomposes in accordance with the rule*

$$[k(\mu)] \times [\ell(\nu)] = [k{+}\ell((\{\mu_s\}^{\langle 2k\rangle} \cdot \{\nu_s\}^{\langle 2\ell\rangle} \cdot B)_N)]. \tag{3.11}$$

where $N = \min(k{+}\ell, n)$ *and*

$$(\lambda)_N = \begin{cases} (\lambda) & \text{if } \lambda'_1 \leq N; \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

Either (3.6) or (3.11) may be used to evaluate tensor products. Equation (3.6) has advantages when a single coefficient $R^{\mu\nu}_\lambda$ is required. In that case signed sequences are not needed. However, equation (3.11) is particularly useful in evaluating complete products.

By way of example, consider the evaluation of the terms, to weight eight, in the tensor product $[2(21)] \times [3(1^2)]$ for the group $SO^*(8)$. Since $N = 4$ all products appearing in (3.11) may be evaluated within $U(4)$ and the signed sequences restricted to the terms

$$\{21_s\}^{\langle 4\rangle} = \{21\} - \{21^3\}, \tag{3.13a}$$

$$\{1^2_s\}^{\langle 6\rangle} = \{1^2\}. \tag{3.13b}$$

Their product in $U(4)$ yields the terms

$$\{21^3\} + \{2^21\} - \{2^31\} + \{31^2\} + \{32\} - \{321^2\}. \tag{3.14}$$

Since we are evaluating terms to weight eight only terms in the B-series to weight three are relevant, that is the two terms

$$\{0\} + \{1^2\}, \tag{3.15}$$

and forming the product we obtain the terms, to weight eight, as

$$\begin{aligned} &\{21^3\} + \{2^21\} + \{2^31\} + \{31^2\} + \{32\} + 3\{321^2\} \\ &+ 2\{32^2\} + 2\{3^21\} + \{41^3\} + 2\{421\} + \{43\}. \end{aligned} \tag{3.16}$$

Changing the notation to that for $SO^*(8)$ and inserting the integer 5 in front of each partition, we finally obtain the result

$$\begin{aligned} [2(21)] \times [3(1^2)] =& [5(21^3)] + [5(2^21)] + [5(2^31)] + [5(31^2)] + [5(32)] \\ &+ 3[5(321^2)] + 2[5(32^2)] + 2[5(3^21)] + [5(41^3)] \\ &+ 2[5(421)] + [5(43)] + \cdots. \end{aligned} \tag{3.17}$$

4. The explicit decomposition of $H \times H$

Equation (3.11) is also useful in deriving explicit complete fomulae for tensor products. In particular the use of (3.11) immediately leads to the result

$$H_m \times H_{m'} = [1(m)] \times [1(m')] = \sum_{p=0}^{\infty} \sum_{x=0}^{min(m,m')} [2(m + m' + p - x, p + x)]. \tag{4.1}$$

This can be seen by noting that successive multiplication in $U(2)$ of a term $\{p^2\}$ in B by $\{m\}$ and then $\{m'\}$ can be carried out diagrammatically to give:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline a & a & a & c & c & c & c & c & c & d & d & d \\ \hline b & b & b & d & d \\ \cline{1-5} \end{array} \tag{4.2}$$

where there are precisely p columns containing the pair (a, b), m entries c in the first row and m' entries d, x of which are in the second row and the remainder in the first, with no identical entries d allowed in the same column.

Extending this analysis to the case of the square of the basic harmonic representation we have

Proposition 4.1 *For H as defined in (1.1)*

$$H^2 = H \times H = \sum_{r=0}^{\infty} \sum_{s=0}^{r} (r - s + 1)(s + 1)[2(r, s)]. \tag{4.3}$$

Proof We have from (1.1) and (4.1)

$$\begin{aligned} H^2 = H \times H &= \sum_{m,m'=0}^{\infty} [1(m)] \times [1(m')] \\ &= \sum_{p,m,m'=0}^{\infty} \sum_{x=0}^{min(m,m')} [2(m + m' + p - x, p + x)] \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{r} C_{r,s} \; [2(r, s)], \end{aligned} \tag{4.4}$$

where $C_{r,s}$ is the number of diagrams of type (4.2) having rows of length r and s for any p, m, m' and x. For fixed r and s it is clear that the distribution of the letters is such that the number of d's in the second row of length s can vary from 0 to s, while the number in the first row of length r can, independently, vary from 0 to $r-s$. Thus $C_{r,s} = (s+1)(r-s+1)$, as required.

It is useful for later work to split H into its even and odd parts, H_+ and H_-, respectively, which are defined in (1.2). The coefficients $C_{rs}^{\eta\zeta}$ of the terms $[2(r,s)]$ in the various products $H_\eta \times H_\zeta$ are given by the following proposition

Proposition 4.2 *Let* $C_{rs} = (r-s+1)(s+1)$. *Then for* $\eta, \zeta \in \{+,-\}$ *we have*

$$H_\eta \times H_\zeta = \sum_{r=0}^{\infty} \sum_{s=0}^{r} \ C_{rs}^{\eta\zeta} \ [2(r,s)] \tag{4.5}$$

with

$$C_{rs}^{++} = \begin{cases} \frac{1}{2}(C_{rs}+1) & \text{if } r \text{ and } s \text{ are both even;} \\ \frac{1}{2}C_{rs} & \text{if } r \text{ and } s \text{ are both odd;} \\ 0 & \text{otherwise,} \end{cases} \tag{4.6a}$$

$$C_{rs}^{--} = \begin{cases} \frac{1}{2}(C_{rs}-1) & \text{if } r \text{ and } s \text{ are both even;} \\ \frac{1}{2}C_{rs} & \text{if } r \text{ and } s \text{ are both odd;} \\ 0 & \text{otherwise,} \end{cases} \tag{4.6b}$$

$$C_{rs}^{+-} = C_{rs}^{-+} = \begin{cases} \frac{1}{2}C_{rs} & \text{if } r \text{ is even and } s \text{ is odd;} \\ \frac{1}{2}C_{rs} & \text{if } r \text{ is odd and } s \text{ is even;} \\ 0 & \text{otherwise.} \end{cases} \tag{4.6c}$$

Proof First it should be noted that

$$H_\eta \times H_\zeta = [2(M_\eta \cdot M_\zeta \cdot B)_2], \tag{4.7}$$

where $M_+ = \sum_{m \text{even}} \{m\}$ and $M_- = \sum_{m \text{odd}} \{m\}$. Since all the term of B are of even weight it follows that all terms $[2(r,s)]$ of both H_+^2 and H_-^2 must be of even weight, so that r and s are either both even or both odd. Similarly, all the terms of $H_+ \times H_-$ must have r and s of opposite parity. This accounts for all the 0's appearing in (4.6a-c).

Separating $H^2 = (H_+ + H_-)^2 = (H_+^2 + H_-^2) + 2(H_+ \times H_-)$, as given by (4.3), into terms of even and odd weight, then immediately gives (4.6c). Moreover to separate the terms of $H_+^2 + H_-^2$ into those of (4.6a) and (4.6b) it is merely necessary to show that $H_+^2 - H_-^2 = \sum_{r=0}^{\infty} \sum_{s=0}^{r} [2(r,s)]$ with the summation restricted to r and s both even. This may be established, by using (4.7) and various S-function series identities[5,13] which imply that

$$H_+^2 - H_-^2 = [2((M_+^2 - M_-^2) \cdot B)_2] = [2(W \cdot B)_2] = [2(D_2)], \tag{4.8}$$

where the restriction of the S-function series D to two-part partitions gives $D_2 = \sum_r^{\infty} \sum_{s=0}^{r} \{r,s\}$ with r and s both even, as required.

5. Symmetrised powers of irreducible representations of $SO^*(2n)$

Following the techniques of Section 6 of KW2 it is not difficult to derive the following general formula for symmetrised powers or plethysms of arbitrary irreducible representations of $SO^*(2n)$:

Proposition 5.1 *Let the partition λ be such that $\lambda_1' \leq \min(k,n)$ and let ρ be an arbitrary partition of r, then*

$$[k(\lambda)] \otimes \{\rho\} = \sum_{\mu} y^{\mu}_{\lambda\rho}\ [kr(\mu)] \tag{5.1}$$

where the summation is over all partitions μ such that $\mu_1' \leq \min(kr,n)$ and the coefficients $y^{\mu}_{\lambda\rho}$ are determined by the expansion

$$\left(\left(\{\lambda_s\}^{\langle 2k\rangle} \cdot B\right) \otimes \{\rho\}\right) \cdot A = \sum_{\mu} y^{\mu}_{\lambda\rho}\ \{\mu_s\}^{\langle 2kr\rangle} \tag{5.2}$$

where $A = B^{-1}$.

Furthermore, just as for ease of calculation (2.8) can be replaced by (2.10), so (5.2) can be replaced by

$$\left(\left(\left(\{\lambda_s\}^{\langle 2k\rangle}_N \cdot B_N\right)_N \otimes \{\rho\}\right)_M \cdot A_M\right)_M = \sum_{\mu} y^{\mu}_{\lambda\rho}\ \{\mu_s\}^{\langle 2kr\rangle}_M \tag{5.3}$$

where $N = \min(2k,n)$ and $M = \min(2kr,n)$. Finally in order to read off the required plethysm coefficients in (5.1) from (5.3) it is only necessary to retain the leading term $\{\mu\}$ in each signed sequence $\{\mu_s\}^{\langle 2kr\rangle}_M$, since it is only the leading term of each signed sequence that satisfies the required $Sp(2kr)$-standardness condition $\mu_1' \leq kr$. This implies that in using (5.3) in (5.1) we may effectively replace $M = \min(2kr,n)$ by $\min(kr,n)$, a considerable simplification which leads to:

Corollary 5.2 *With the notation of Proposition 5.1 the $SO^*(2n)$ plethysm coefficients $y^{\mu}_{\lambda\rho}$ are determined by*

$$\left(\left(\left(\{\lambda_s\}^{\langle 2k\rangle}_N \cdot B_N\right)_N \otimes \{\rho\}\right)_K \cdot A_K\right)_K = \sum_{\mu} y^{\mu}_{\lambda\rho}\ \{\mu\} \tag{5.4}$$

where $N = \min(2k,n)$ and $K = \min(kr,n)$.

The significance of (5.3) and the subsequent remarks leading finally to (5.4) can be seen in the evaluation of the terms in the plethysm $[2(21)] \otimes \{21\}$ of $SO^*(24)$. In such a case we have $k = 2$, $r = 3$ and $n = 12$ so that $N = 4$, $M = 12$ and $K = 6$. We show how to calculate all terms $[6(\mu)]$ of $[2(21)] \otimes \{21\}$ up to weight 16 and of width $\mu_1 \leq 3$ using (5.3). Such terms will necessarily have length $\mu_1' \leq 6$.

Since $k = 2$ the signed sequence is evaluated in $Sp(4)$ giving

$$\{21_s\}^{\langle 4\rangle}_4 = \{21\} - \{21^3\} \tag{5.5}$$

Next the terms in the B-series up to weight 16, width 3 and length 4 are:

$$\begin{array}{lllll} \{0\} & +\{1^2\} & +\{1^4\} & +\{2^2\} & +\{2^21^2\} \\ +\{2^4\} & +\{3^2\} & +\{3^21^2\} & +\{3^22^2\} & +\{3^4\} \end{array} \tag{5.6}$$

The tensor product of the terms in (5.5) with those of (5.6) is to be carried out in $U(N)$ with $N = 4$. Again up to weight 16, width 3 and length 4 this gives

$$\begin{array}{lllll}
\{21\} & +\{2^21\} & +\{31^2\} & +\{32\} & +\{321^2\} \\
+\{32^2\} & +\{3^21\} & +\{3^221\} & &
\end{array} \tag{5.7}$$

Now we calculate the mixed symmetry third order plethysm signified by $\{21\}$ of this sum of terms in $U(M)$ with $M = 12$ to give up to weight 16, width 3 and now length 12:

$$\begin{array}{lllll}
\{2^41\} & +\{2^41^3\} & +2\{2^51\} & +\{2^51^3\} & +2\{2^61\} \\
+\{2^71\} & +\{321^4\} & +2\{32^21^2\} & +3\{32^21^4\} & +\{32^3\} \\
+9\{32^31^2\} & +5\{32^31^4\} & +6\{32^4\} & +15\{32^41^2\} & +4\{32^41^4\} \\
+10\{32^5\} & +11\{32^51^2\} & +7\{32^6\} & +\{3^21^3\} & +2\{3^21^5\} \\
+3\{3^221\} & +12\{3^221^3\} & +7\{3^221^5\} & +18\{3^22^21\} & +33\{3^22^21^3\} \\
+9\{3^22^21^5\} & +45\{3^22^31\} & +40\{3^22^31^3\} & +54\{3^22^41\} & +12\{3^31^2\} \\
+20\{3^31^4\} & +5\{3^31^6\} & +10\{3^32\} & +60\{3^321^2\} & +51\{3^321^4\} \\
+40\{3^32^2\} & +117\{3^32^21^2\} & +71\{3^32^3\} & +32\{3^41\} & +70\{3^41^3\} \\
+120\{3^421\} & +28\{3^5\} & & &
\end{array} \tag{5.8}$$

Then the terms in the A-series up to weight 16, width 3 and length 12 are found to be

$$\begin{array}{lllll}
\{0\} & -\{1^2\} & +\{21^2\} & -\{2^3\} & -\{31^3\} \\
+\{32^21\} & -\{3^22^2\} & +\{3^4\} & &
\end{array} \tag{5.9}$$

and their tensor product with the terms in (5.8) calculated in $U(12)$ gives

$$\begin{array}{lllll}
\{2^41\} & -\{2^41^5\} & +\{2^51\} & -\{2^51^3\} & +\{321^4\} \\
-\{321^6\} & +2\{32^21^2\} & -2\{32^21^6\} & +\{32^3\} & +4\{32^31^2\} \\
-4\{32^31^4\} & -\{32^31^6\} & +3\{32^4\} & -3\{32^41^4\} & +2\{32^5\} \\
-2\{32^51^2\} & +\{3^21^3\} & -\{3^21^7\} & +3\{3^221\} & +5\{3^221^3\} \\
-5\{3^221^5\} & -3\{3^221^7\} & +10\{3^22^21\} & -10\{3^22^21^5\} & +10\{3^22^31\} \\
-10\{3^22^31^3\} & +7\{3^31^2\} & -7\{3^31^6\} & +6\{3^32\} & +13\{3^321^2\} \\
-13\{3^321^4\} & +12\{3^32^2\} & +6\{3^32^3\} & +10\{3^41\} & +14\{3^421\} \\
+4\{3^5\} & & & &
\end{array} \tag{5.10}$$

Now the labelling is changed to that of irreducible representations of $SO^*(24)$ with $kr = 6$ inserted before each partition to yield

$$\begin{array}{lllll}
[6(2^41)] & -[6(2^41^5)] & +[6(2^51)] & -[6(2^51^3)] & +[6(321^4)] \\
-[6(321^6)] & +2[6(32^21^2)] & -2[6(32^21^6)] & +[6(32^3)] & +4[6(32^31^2)] \\
-4[6(32^31^4)] & -[6(32^31^6)] & +3[6(32^4)] & -3[6(32^41^4)] & +2[6(32^5)] \\
-2[6(32^51^2)] & +[6(3^21^3)] & -[6(3^21^7)] & +3[6(3^221)] & +5[6(3^221^3)] \\
-5[6(3^221^5)] & -3[6(3^221^7)] & +10[6(3^22^21)] & -10[6(3^22^21^5)] & +10[6(3^22^31)] \\
-10[6(3^22^31^3)] & +7[6(3^31^2)] & -7[6(3^31^6)] & +6[6(3^32)] & +13[6(3^321^2)] \\
-13[6(3^321^4)] & +12[6(3^32^2)] & +6[6(3^32^3)] & +10[6(3^41)] & +14[6(3^421)] \\
+4[6(3^5)] & & & &
\end{array} \tag{5.11}$$

At first the appearance of negative terms seems disconcerting until it is realised that they correspond to non-standard terms in the signed sequences $\{\mu_s\}^{(12)}$ of (5.3).

Restricting attention, as required, to $SO^*(24)$-standard terms in accordance with (5.4), finally yields the result
$[2(21)] \otimes \{21\} =$

$$\begin{array}{lllll}
[6(2^41)] & +\,[6(2^51)] & +\,[6(321^4)] & +\,2[6(32^21^2)] & +\,[6(32^3)] \\
+\,4[6(32^31^2)] & +\,3[6(32^4)] & +\,2[6(32^5)] & +\,[6(3^21^3)] & +\,3[6(3^221)] \\
+\,5[6(3^221^3)] & +\,10[6(3^22^21)] & +\,10[6(3^22^31)] & +\,7[6(3^31^2)] & +\,6[6(3^32)] \\
+\,13[6(3^321^2)] & +\,12[6(3^32^2)] & +\,6[6(3^32^3)] & +\,10[6(3^41)] & +\,14[6(3^421)] \\
+\,4[6(3^5)] & +\,\cdots & & &
\end{array} \qquad (5.12)$$

up to weight 16 and width 3, where it is to be noted that, as promised, the surviving terms all have length $\mu_1' \leq 6$. This is because $K = \min(kr, n) = 6$. It would clearly have been simpler to use (5.4) at an earlier stage and discard all terms of length greater than 6 in (5.8)-(5.11) rather than to use (5.3) and keep terms up to length 12. At each step the calculation would have involved fewer terms and the final signed sequence problems would have been circumvented. This example, while exhibiting the fact that signed sequences do emerge in a natural way, serves to illustrate the computational merits of Corollary 4.2.

6. Resolution of $H^2 = H \otimes \{2\} + H \otimes \{1^2\}$

In the special case for which ρ is a partition of 2, so that $r = 2$, we have $K = N = \min(2k, n)$ in (5.4). Consequently for symmetrised squares of irreducible representations of $SO^*(2n)$ we have

$$[k(\lambda)] \otimes \{\rho\}] = [\big(\big(\{\lambda_s\}^{\langle 2k\rangle} \cdot B\big) \otimes \{\rho\}\big) \cdot A\big)_N], \qquad (6.1)$$

where all products and plethysms are to be carried out in $U(N)$. Setting $\{\rho\} = \{2\}$ and $\{1^2\}$, using the algebra of plethysms and the fact that[5] $(B \otimes \{2\}) \cdot A = B_+ BA = B_+$ and $(B \otimes \{1^2\}) \cdot A = B_- BA = B_-$ it follows that

$$[k(\lambda)] \otimes \{2\} \;= [2k((\{\lambda_s\}^{\langle 2k\rangle} \otimes \{2\}) \cdot B_+)_N] + [2k((\{\lambda_s\}^{\langle 2k\rangle} \otimes \{1^2\}) \cdot B_-)_N]; \quad (6.2a)$$

$$[k(\lambda)] \otimes \{1^2\} = [2k((\{\lambda_s\}^{\langle 2k\rangle} \otimes \{1^2\}) \cdot B_+)_N] + [2k((\{\lambda_s\}^{\langle 2k\rangle} \otimes \{2\}) \cdot B_-)_N], \quad (6.2b)$$

where $N = min(2k, n)$.

Further specialisation of the above result leads to

$$H_m \otimes \{2\} \;= [1(m)] \otimes \{2\} = \sum_{p=0}^{\infty} \sum_{x=0}^{m} [2(2m + p - x, p + x)] \text{ with } p{+}x \text{ even}; \qquad (6.3a)$$

$$H_m \otimes \{1^2\} = [1(m)] \otimes \{1^2\} = \sum_{p=0}^{\infty} \sum_{x=0}^{m} [2(2m + p - x, p + x)] \text{ with } p{+}x \text{ odd}. \qquad (6.3b)$$

This can be seen by noting that in $U(2)$ each term $\{p^2\}$ of B belongs to B_+ or B_- according as p is even or odd, respectively, while $\{m\} \otimes \{2\}$ and $\{m\} \otimes \{1^2\}$ contain terms of the form $\{2m - x, x\}$ with x even and odd, respectively. Typical terms contributing to (6.3a) are represented diagrammatically by:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
a & a & a & a & c & c & c & c & c & d & d & d \\
\hline
b & b & b & b & d & d \\
\cline{1-6}
\end{array} \qquad (6.4)$$

where there are precisely p columns containing the pair (a,b), x columns containing the pair (c,d), $m-x$ columns containing just c and the same number containing just d, with p and x either both even or both odd.

Summing over all m we obtain

$$\sum_{m=0}^{\infty} H_m \otimes \{2\} = \sum_{r=0}^{\infty}\sum_{s=0}^{r} (s+1)[2(r,s)] \quad \text{with } r \text{ and } s \text{ both even;} \tag{6.5a}$$

$$\sum_{m=0}^{\infty} H_m \otimes \{1^2\} = \sum_{r=0}^{\infty}\sum_{s=0}^{r} (s+1)[2(r,s)] \quad \text{with } r \text{ and } s \text{ both odd.} \tag{6.5a}$$

The first of these follows from the fact that for fixed r and s the distribution of letters in diagrams of type (6.4) is such that the number of d's in the second row of length s can vary from 0 to s, while the number in the first row is necessarily $(r-s)/2$. The second follows in the same way. The only difference is now that instead of r and s both being even, they are both odd.

This allows us to resolve H^2 into its symmetric and antisymmetric parts $H \otimes \{\rho\}$ with $\{\rho\} = \{2\}$ and $\{1^2\}$, respectively:

Proposition 6.1 *Let* $C_{rs} = (r-s+1)(s+1)$. *Then for* $\{\rho\} = \{2\}$ *and* $\{1^2\}$ *we have*

$$H \otimes \{\rho\} = \sum_{r=0}^{\infty}\sum_{s=0}^{r} C_{rs}^{\{\rho\}}\ [2(r,s)] \tag{6.6}$$

with

$$C_{rs}^{\{2\}} = \begin{cases} \frac{1}{2}(C_{rs}+1) & \text{if } r \text{ and } s \text{ are both even;} \\ \frac{1}{2}(C_{rs}-1) & \text{if } r \text{ and } s \text{ are both odd;} \\ \frac{1}{2}C_{rs} & \text{otherwise,} \end{cases} \tag{6.7a}$$

$$C_{rs}^{\{1^2\}} = \begin{cases} \frac{1}{2}(C_{rs}-1) & \text{if } r \text{ and } s \text{ are both even;} \\ \frac{1}{2}(C_{rs}+1) & \text{if } r \text{ and } s \text{ are both odd;} \\ \frac{1}{2}C_{rs} & \text{otherwise,} \end{cases} \tag{6.7b}$$

Proof Since

$$H^2 = \left(\sum_m H_m\right)^2 = \sum_m H_m^2 + \sum_{m\neq m'} H_m H_{m'}, \tag{6.8}$$

it follows that

$$\begin{aligned} H \otimes \{2\} &= \left(\sum_m H_m\right) \otimes \{2\} = \sum_m \left(H_m \otimes \{2\}\right) + \frac{1}{2}\sum_{m\neq m'} H_m H_{m'} \\ &= \sum_m \left(H_m \otimes \{2\}\right) + \frac{1}{2}\left(H^2 - \sum_m H_m^2\right) \\ &= \frac{1}{2}\left(H^2 + \sum_m \left(H_m \otimes \{2\} - H_m \otimes \{1^2\}\right)\right). \end{aligned} \tag{6.9a}$$

Similarly

$$H \otimes \{1^2\} = \frac{1}{2}\Big(H^2 - \sum_m \Big(H_m \otimes \{2\} - H_m \otimes \{1^2\}\Big)\Big). \tag{6.9b}$$

The results (6.7) then follow from (4.3) and (6.5).

7. Relations between group chains and irreducible representations of $SO^*(2n)$ and $Sp(2n\Re)$

Starting with the metaplectic group $Mp(4nk)$ we may relate the decompositions involving the non-compact subgroups $SO^*(2n)$ and $Sp(2n,R)$ by means of the commutative diagram:

$$\begin{array}{ccccc}
SO^*(2n)\times Sp(2k) & \longleftarrow & Mp(4nk) & \longrightarrow & Sp(2n,R)\times O(2k) \\
\downarrow & & & & \downarrow \\
U(n)\times Sp(2k) & & & & U(n)\times O(2k) \\
\downarrow & & & & \downarrow \\
U(n)\times SO(2k) & \longrightarrow & & \longleftarrow & U(n)\times SO(2k)
\end{array} \tag{7.1}$$

The terminal subgroup in each case is $U(n)\times SO(2k)$. Taking into account the labels used to distinguish mutually associate pairs of irreducible representations of $Sp(2n\Re)$, the decomposition of the metaplectic irreducible representation $\tilde\Delta$ of $Mp(4nk)$ proceeds as indicated below:

$$\begin{array}{ccccc}
\sum_\kappa [k(\kappa)]\times\langle\kappa\rangle & \longleftarrow & \tilde\Delta & \longrightarrow & \sum_\lambda \langle k(\lambda)\rangle\times[\lambda] \\
\downarrow & & & & \downarrow \\
\sum_\kappa [k(\kappa)]_{U(n)}\times\langle\kappa\rangle & & & & \sum_\lambda \langle k(\lambda)\rangle_{U(n)}\times[\lambda] \\
\downarrow & & & & \downarrow \\
\sum_\kappa [k(\kappa)]_{U(n)}\times[\kappa/AD] & \longrightarrow & & \longleftarrow & \sum_\lambda \big(\langle k(\lambda+(1-\delta_{\lambda'_1 k})\lambda^*)\big)\rangle_{U(n)}\times[\lambda]
\end{array} \tag{7.2}$$

where the symbols $[\cdots]_{U(n)}$ and $\langle\cdots\rangle_{U(n)}$ signify restriction from $SO^*(2n)$ and $Sp(2n,\Re)$, respectively, to $U(n)$, while the skew products of κ with A and D correspond to passing from $Sp(2k)$ up to $U(2k)$ and then down to $SO(2k)$. It should be noted that at the level of $U(n)\times SO(2k)$ the summations over both κ and λ are restricted so that these partitions have no more than P parts with $P=\min(k,n)$.

Since[13]

$$AD = W = \sum_{r=0}^{\infty}\sum_{s=0}^{r} (-1)^s\{r,s\} \quad \text{with } r-s \text{ even}, \tag{7.3}$$

it follows that on comparing the terms of the form $\cdots \times [\lambda]$ we have

$$[k(\lambda \cdot W)]_{U(n)} = \langle k(\lambda)\rangle_{U(n)} + \left(1 - \delta_{\lambda_1' k}\right)\langle k(\lambda^*)\rangle_{U(n)}. \tag{7.4}$$

As special cases of this with $k = 1$ and $\lambda = (0)$ and (1), we obtain:

$$\left(H_+\right)_{U(n)} = [1(M_+)]_{U(n)} = \langle 1(0)\rangle_{U(n)} + \langle 1(0^*)\rangle_{U(n)}; \tag{7.5a}$$

$$\left(H_-\right)_{U(n)} = [1(M_-)]_{U(n)} = \langle 1(1)\rangle_{U(n)}. \tag{7.5b}$$

It should be stressed that quite generally a knowledge of the restriction to $U(n)$ of any direct sum of harmonic series unitary irreducible representations of both $SO^*(2n)$ and $Sp(2n,\Re)$ is sufficient to determine these representations up to equivalence. This is because such representations are determined up to equivalence by their characters which are themselves evaluated on elements of the maximal compact subgroup $U(n)$.

8. Powers and plethysms of irreducible representations of $SO^*(2n)$ from those of $Sp(2n,\Re)$

The results of the previous section lead to an alternative method of computing powers of the basic harmonic representation H of $SO^*(2n)$ and its constituents H_+ and H_-. Since $H = H_+ + H_-$ it follows from (7.5) that the $U(n)$ content of the harmonic representation H of $SO^*(2n)$ coincides with that of the representation S of $Sp(2n,\Re)$, where

$$S = \langle 1(0)\rangle + \langle 1(0^*)\rangle + \langle 1(1)\rangle. \tag{8.1}$$

The same must be true of both their powers and plethysms.

Since S is a self-associate representation of $Sp(2n,\Re)$ it follows that its pth power may be written in the form:

$$S^p = \sum_{\mu:\mu_1'\leq P} g^\mu \ \langle p(\mu + (1-\delta_{\mu_1' p})\mu^*\rangle, \tag{8.2}$$

with $P = \min(p,n)$, for some set of coefficients g^μ. It then follows from (7.5) that

$$H^p = \sum_{\mu:\mu_1'\leq P} g^\mu \ [p((\mu \cdot W)_P)] \tag{8.3}$$

where the subscript P on $(\mu \cdot W)_P$ indicates that the only terms $(\nu)_P$ to be retained are those for which $\nu_1' <= P = \min(p,n)$. Likewise, for any partition π of p, the corresponding pth-fold symmetrised power may be expanded in the form

$$S \otimes \{\pi\} = \sum_{\mu:\mu_1'\leq P} h^\mu \ \langle p(\mu + (1-\delta_{\mu_1' p})\mu^*\rangle, \tag{8.4}$$

for some particular set of coefficients h^μ. It then follows that

$$H \otimes \{\pi\} = \sum_{\mu:\mu_1'\leq P} h^\mu \ [p((\mu \cdot W)_P)]. \tag{8.5}$$

A similar situation applies to H_+ and H_-, and indeed to plethysms and powers of any sum of $SO^*(2n)$ representations of the form $[k(\lambda \cdot W)]$ as in (7.4). Conversely, plethysms and powers of any self-associate sum of $Sp(2n,\Re)$ representations of the form $\langle k(\kappa \cdot V)\rangle$ can be evaluated from a knowledge of the powers and plethysms of $[k(\kappa)]$ in $SO^*(2n)$, where $V = W^{-1} = W'$.

As a final example we compute the terms of $H_- \otimes \{21\}$ up to weight 12 and width 4 in $SO^*(24)$ starting from the $Sp(24,\Re)$ plethysm

$\langle 1(1)\rangle \otimes \{21\} =$

$$\begin{array}{lllll} \langle 3(21)\rangle & + \langle 3(21^3)\rangle & + \langle 3(2^21)\rangle & + 2\langle 3(31^2)\rangle & + 2\langle 3(32)\rangle \\ + 2\langle 3(321^2)\rangle & + 2\langle 3(32^2)\rangle & + 3\langle 3(3^21)\rangle & + 2\langle 3(41)\rangle & + 2\langle 3(41^3)\rangle \\ + 6\langle 3(421)\rangle & + 4\langle 3(43)\rangle & + 4\langle 3(431^2)\rangle & + 5\langle 3(432)\rangle & + 5\langle 3(4^21)\rangle \\ + 2\langle 3(4^23)\rangle & & & & \end{array} \tag{8.6}$$

Now we remove the prefix $p = 3$ and standardise the irreducible representations in the group $U(3)$ to give

$$\begin{array}{lllll} \{21\} & + \{2^21\} & + 2\{31^2\} & + 2\{32\} & + 2\{32^2\} \\ + 3\{3^21\} & + 2\{41\} & + 6\{421\} & + 4\{43\} & + 5\{432\} \\ + 5\{4^21\} & + 2\{4^23\} & & & \end{array} \tag{8.7}$$

The terms in the W-series up to weight 12 are

$$\begin{array}{lllll} \{0\} & - \{1^2\} & + \{2\} & + \{2^2\} & - \{31\} \\ - \{3^2\} & + \{4\} & + \{42\} & + \{4^2\} & \end{array} \tag{8.8}$$

Forming the tensor product, in $U(3)$, of (8.7) and (8.8), and keeping terms up weight 12 and width 4 gives

$$\begin{array}{lllll} \{21\} & + \{2^21\} & + 2\{31^2\} & + 2\{32\} & + 2\{32^2\} \\ + 4\{3^21\} & + \{3^3\} & + 3\{41\} & + 7\{421\} & + 6\{43\} \\ + 9\{432\} & + 7\{4^21\} & + 6\{4^23\} & & \end{array} \tag{8.9}$$

These $U(3)$ irreducible representations are now converted into $SO^*(24)$ irreducible representations by inserting $p = 3$ before each partition and adopting the notation appropriate to the group $SO^*(24)$ leading to

$H_- \otimes \{21\} =$

$$\begin{array}{lllll} [3(21)] & + [3(2^21)] & + 2[3(31^2)] & + 2[3(32)] & + 2[3(32^2)] \\ + 4[3(3^21)] & + [3(3^3)] & + 3[3(41)] & + 7[3(421)] & + 6[3(43)] \\ + 9[3(432)] & + 7[3(4^21)] & + 6[3(4^23)] & + \cdots . & \end{array} \tag{8.10}$$

This is the same as the result that can be found using (5.1) and (5.4). While the above calculation was carried out for $SO^*(24)$ it should be noted that the result is valid for all $SO^*(2n)$ with $n \geq 6$.

9. Concluding remarks

The objective in writing this paper has been to complete the results hinted at in KW1. In the process methods for calculating tensor products and plethysms of the infinite dimensional unitary irreducible representations of $SO^*(2n)$ have been developed. In addition explicit results have been obtained for the case of the basic infinite-dimensional harmonic representation H of $SO^*(2n)$ and its various constituents $H_\pm$ and H_m. Furthermore, the embedding of the product groups $SO^*(2n) \times Sp(2k)$ and $Sp(2n, \Re) \times O(2k)$ in the metaplectic group $Mp(4nk)$ has been shown to yield interesting and indeed useful relationships between irreducible representations of the non-compact groups $SO^*(2n)$ and $Sp(2n, \Re)$ as well as their powers and plethysms.

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