SYMPLECTIC MODELS OF $n-$PARTICLE SYSTEMS

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The universe is infinite in all directions, not only above us in the large but also below us in the small
— Emil Wiechert (1896)

Abstract

The dynamical group $\mathcal{S}p(6n, R)$ is used to give a description of the states for $n-$noninteracting particles confined by an isotropic three-dimensional harmonic oscillator potential. The subgroup structure of the dynamical group is used to determine the relevant spins and unitary group representations of the $n-$particle states. This is a necessary precursor to developing model Hamiltonians for describing systems such as quantum dots and nuclei in terms of polynomials in the dynamical group generators.

1. Introduction

The isotropic three-dimensional harmonic oscillator (henceforth we will abbreviate to just $\mathcal{HO}$) for a single particle is one of the few problems whose Schrodinger equation is completely solvable. The complete set of states span a single irreducible representation of the metaplectic group $\mathcal{M}p(6)$ which is the covering group of the symplectic group $\mathcal{S}p(6, R)$[1-8]. Upon the restriction $\mathcal{M}p(6) \to \mathcal{S}p(6, R)$ the single irreducible representation of $\mathcal{M}p(6)$ decomposes into a pair of irreducible representations which we designate as $\langle s; (0)\rangle$ and $\langle s; (1)\rangle$ [3]. The irreducible representation $\langle s; (0)\rangle$ is spanned by the complete set of *even* parity states and $\langle s; (1)\rangle$ by the odd parity states. Throughout this paper we shall often just write $\mathcal{S}p(N)$ rather than $\mathcal{S}p(N, R)$ with the understanding that we will always be referring to the non-compact symplectic group defined on reals and *not* the compact symplectic group.

For $n-$noninteracting particles the dynamical group is $\mathcal{M}p(6n)$[7,8] and again the complete set of states span a single irreducible representation of $\mathcal{M}p(6n)$. Upon the restriction $\mathcal{M}p(6n) \to \mathcal{S}p(6n, R)$ the single irreducible representation of $\mathcal{M}p(6n)$ decomposes into a pair of irreducible representations which again we designate as $\langle s; (0)\rangle$ and $\langle s; (1)\rangle$ with the *even* parity states spanning the $\langle s; (0)\rangle$ irreducible representation and $\langle s; (1)\rangle$ by the odd parity states.

The group $\mathcal{M}p(6n)$ has a very rich subgroup structure[4,5,7,8] which we will first outline and then direct our attention to the relevant group-subgroup decompositions leading to a detailed classification of the states and the identification of their spin and unitary $\mathcal{U}(3)$ structure. This should then make it possible to start to develop model Hamiltonians in terms of polynomials in the relevant group generators for $n-$interacting particles in applications associated with quantum dots and symplectic models of nuclei.

2. The substructure of the dynamical group $\mathcal{M}p(6n)$

Let us start by considering the slightly more general case of $n-$noninteracting particles in a $d-$dimensional harmonic oscillator potential. The dynamical group may be formally constructed from the coordinate and momentum operators of the individual particles under the usual Heisenberg commutation relations. Bilinear combinations of these operators are constructed to close under commutation and the associated Lie algebra identified. It is readily found that indeed they close upon the algebra associated with the metaplectic group $\mathcal{M}p(2nd)$ which is the covering group of the non-compact symplectic group $\mathcal{S}p(2nd, R)$. The metaplectic group $\mathcal{M}p(2nd)$ has a very rich subgroup structure[8] as shown in Fig.1. These subgroup structures can be determined by contracting on particle or spatial indices. The diversity of the subgroup structures reflect different ways of separating the spatial and particle number dependencies. Thus the subgroup $\mathcal{O}(d)$ describes the angular momentum states of the system while the subgroup $\mathcal{O}(n)$ gives information on the permutational symmetries of the states via the symmetric group $\mathcal{S}(n)$ which is a subgroup of $\mathcal{O}(n)$.

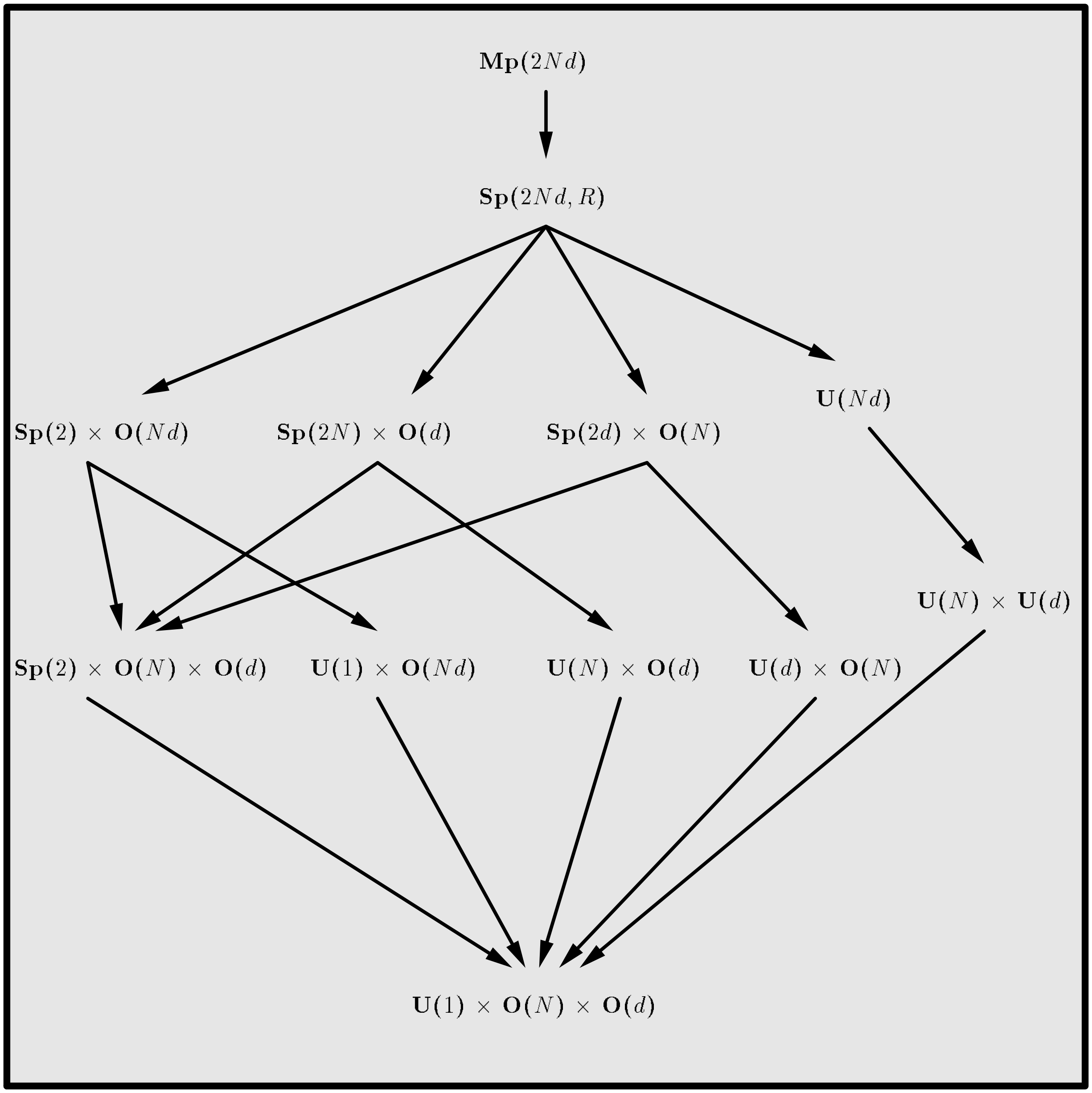

Fig. 1: Group-subgroup structures appropriate to quantum dots.

3. Labelling $\mathcal{S}p(2N,R)$ irreducible representations

The labelling of the irreducible representations of compact Lie groups in terms of partition labels is well established[9]. Here we shall limit ourselves to discussion of the so-called positive discrete unitary irreducible representations of the group $\mathcal{S}p(2n,R)$ and its double covering group, $\mathcal{M}p(2n)$, drawing heavily upon references [2] and [3]. These irreducible representations are all infinite dimensional and are characterised by a *lowest weight* with respect to the ordering of weights of the maximal compact subgroup $U(n)$. There exists a harmonic representation, $\tilde{\Delta}$, associated with the Heisenberg algebra. This is a true, unitary, infinite dimensional irreducible representation of the double covering group $\mathcal{M}p(2n)$ of $\mathcal{S}p(2n,R)$, the so-called *metaplectic group*. This representation is reducible into the sum of two irreducible representations $\tilde{\Delta}_+$ and $\tilde{\Delta}_-$ whose leading weights are $(\frac{1}{2}\frac{1}{2}\dots\frac{1}{2})$ and $(\frac{3}{2}\frac{1}{2}\dots\frac{1}{2})$ corresponding to the highest weights of the representations $\varepsilon^{\frac{1}{2}}\{0\}$ and $\varepsilon^{\frac{1}{2}}\{1\}$ which appear in the restriction of $\mathcal{S}p(2n,R)$ to its maximal compact subgroup $U(n)$.

The tensor powers $\tilde{\Delta}^k$ all decompose into a direct sum of unitary irreducible representations of $\mathcal{M}p(2n)$. All those irreducible representations which derive from $\tilde{\Delta}^k$ for some k will be referred to as *harmonic series representaions.* All those irreducible representations that appear in $\tilde{\Delta}^k$ will be labelled by the symbols $\langle\frac{k}{2}(\lambda)\rangle$. The harmonic series representations appearing in $\tilde{\Delta}^k$ are in one-to-one correspondence with the terms arising in the branching rule appropriate to the restriction from $\mathcal{M}p(2nk)$ to $\mathcal{S}p(2n,R)\times\mathcal{O}(k)$

$$\tilde{\Delta}\rightarrow\sum_{\lambda}\langle\frac{k}{2}(\lambda)\rangle\times[\lambda] \tag{1}$$

where the summation is carried out over all partitions $(\lambda)=(\lambda_1,\lambda_2,\ldots)$ for which the conjugate partition $(\tilde{\lambda})=(\tilde{\lambda}_1,\tilde{\lambda}_2,\ldots)$ satisfies the constraints

$$\tilde{\lambda}_1+\tilde{\lambda}_2\leq k \tag{2a}$$

and

$$\tilde{\lambda}_1\leq n \tag{2b}$$

Irreducible representations of $\mathcal{S}p(2n,R)$ $\langle\frac{1}{2}k(\lambda)\rangle$ satisfying Eq.(2) will be said to be *standard* and we may limit our attention to these irreducible representations of $\mathcal{S}p(2n,R)$.

The value of $\frac{k}{2}$ maybe an integer (k *even*) or a half-odd-integer (k *odd*). In terms of inputting and outputting $\mathcal{S}p(2n,R)$ labelled irreducible representations into SCHUR it is useful to introduce the equivalent notation

$$\langle s\kappa;(\lambda)\rangle\equiv\langle\frac{k}{2}(\lambda)\rangle \tag{3}$$

where

$$\frac{k}{2}=s+\kappa \tag{4}$$

with κ being the integer part of $\frac{k}{2}$ and the residue part is $s=0$ or $\frac{1}{2}$. Thus we have the typical notational equivalences

$$\langle s1;(\lambda)\rangle\equiv\langle\frac{3}{2}(\lambda)\rangle, k=3 \quad \langle 1;(\lambda)\rangle\equiv\langle 1(\lambda)\rangle \quad k=2$$

SCHUR accepts irreducible representation labels in the form of lists of $\langle s\kappa;\lambda\rangle$ and standardises the input in accordance with the constraints of Eq.(2) making null all non-standard $\mathcal{S}p(2n,R)$ irreducible representations.

4. Lowest energy states for non-interacting particles in a $\mathcal{HO}$

In the case of n non-interacting spin $\frac{1}{2}$ particles in a three-dimensional isotropic $\mathcal{HO}$ potential the energy of a given state is simply the sum of the one-particle energies (cf. Fig. 2) and hence the lowest energy state associated with a given $\mathcal{S}p(6,R)$ multiplet $[\kappa(\lambda)\rangle$ is, relative to the groundstate energy,

$$w_\lambda\hbar\omega \tag{5}$$

where ω is the oscillator angular frequency and w_λ is the weight of the partition (λ). Representations of $\mathcal{S}p(6,R)$ having different partitions but of the same weight will have the same zero-order energy as given in Eq. (5).

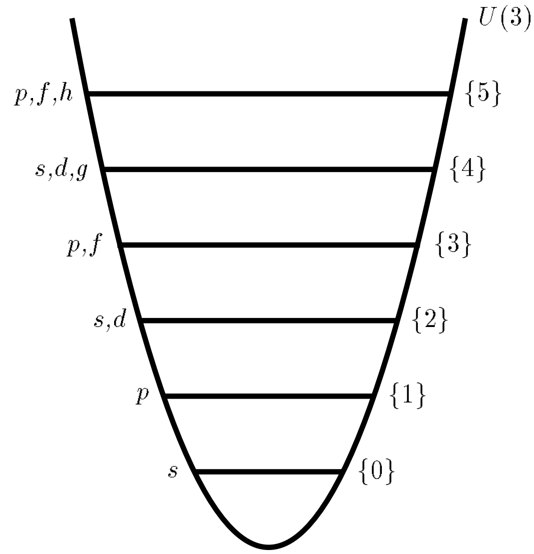

Fig. 2: The states of a single particle in a harmonic oscillator potential.

The states of $n-$particles may be associated with occupations of particles in various of the single particle $\mathcal{U}(3)$ multiplets subject to the Pauli exclusion principle. It is convenient to speak of $n-$particle configurations of the form

$$\{0\}^{m_0}\{1\}^{m_1}\{2\}^{m_2}\ldots \tag{6}$$

where the exponents are the occupation numbers for the various $\mathcal{U}(3)$ single particle irreducible representations. The $\mathcal{U}(3)$ states of weight w for $n-$particles may be determined as follows

1. Partition the integer w into n parts allowing zero parts if necessary.
2. Even weight partitions involve even parity states otherwise odd parity states.
3. Replace each part ,i, by $\{i\}$ which then labels the $\mathcal{U}(3)$ irrep for a single particle in the $i-$th harmonic oscillator orbital. A given orbital i can accommodate up to $d_i = (i+1)(i+2)$ particles with spin $\frac{1}{2}$ and hence partitions having parts, i, with a multiplicity exceeding d_i must be discarded.
4. For a given partition containing k distinct non-repeating parts form the $\mathcal{SU}(2)\times\mathcal{U}(3)$ Kronecker product

$$\{\frac{1}{2}\}\times\{i_1\}.\{\frac{1}{2}\}\times\{i_2\}\cdots.\{\frac{1}{2}\}\times\{i_k\} \tag{7}$$

   to give a series of $\mathcal{SU}(2)^S\times\mathcal{U}(3)$ multiplets.
5. If the parts i are repeated with a multiplicity m then evaluate the plethysm

$$(\{\frac{1}{2}\}\{i\})\otimes\{1^m\} = \sum_{a=[\frac{m+1}{2}]}^{m} {}^{(2a-m+1)}(\{i\}\otimes\{2^{m-a}1^{2a-m}\}) \tag{8}$$

   where the spin multiplicity $(2S+1) = (2a-m+1)$ has been written as a superscript.

   For $n=3$ we have for weight 4 the four partitions

$$4+0+0, \quad 3+1+0, \quad 2+2+0, \quad 2+1+1 \tag{9}$$

Applying the above algorithm we find for the first partition a $\mathcal{U}(3)$ multiplet $\{4\}$ with $S=\frac{1}{2}$ corresponding to two particles in the $\{0\}$ orbital and one in the $\{4\}$ orbital.. The second partition gives two $\mathcal{U}(3)$ multiplets, $\{4\}+\{31\}$ with spins $S=\frac{1}{2}$ and $S=\frac{3}{2}$. These are associated with the states arising from the $\mathcal{U}(3)$ configuration $\{0\}^1\{1\}^1\{3\}^1$ The third partition yields the $\mathcal{U}(3)$ multiplet $\{31\}$ with $S=\frac{3}{2}$ and the

$\mathcal{U}(3)$ multiplets $\{4\}+\{31\}+\{2^2\}$ with spin $S=\frac{1}{2}$, corresponding to the configuration $\{0\}^1\{2\}^2$. The fourth partition yields the two $\mathcal{U}(3)$ multiplets $\{31\}+\{2^2\}$ with spin $S=\frac{3}{2}$ and the three $\mathcal{U}(3)$ multiplets $\{4\}+2\{31\}+\{2^2\}+\{21^2\}$ with spin $S=\frac{1}{2}$, corresponding to the configuration $\{1\}^2\{2\}^1$. Thus for spin $S=\frac{3}{2}$ we obtain the $\mathcal{U}(3)$ multiplets $\{4\}+3\{31\}+\{21^2\}$ and for spin $S=\frac{1}{2}$ the $\mathcal{U}(3)$ multiplets $4\{4\}+4\{31\}+2\{2^2\}+\{21^2\}$.

5. The Lowest $\mathcal{U}(3)$ Multiplets

Filling the first k shells with particles will involve a total of

$$N_k=\frac{k(k+1)(k+2)}{3} \tag{10}$$

particles. If $n-N_k$ particles are in the lowest unfilled shell then the weights w_λ of admissible partitions labelling irreducible representations of $\mathcal{U}(3)$ will be given by

$$w_\lambda=k\left[n-\frac{(k+1)(k+2)(k+3)}{12}\right] \tag{11}$$

Thus for 12 particles we would have $w_\lambda=14$.

If the first k shells are fully occupied then the resultant state has spin $S=0$ and belongs to the $\mathcal{U}(3)$ irrep $\{p,p,p\}$ where

$$p=\left[\frac{(k-1)k(k+1)(k+2)}{2}\right] \tag{12}$$

6. Lowest Energy Even Parity 12-particle States

It is desirable to consider a reasonably large number of particles to bring out the main features of the $n-$particle problem. To be specific I shall consider the case of $12-$particles and initially just the even parity states. The lowest states will occur with the first two shells fully occupied and the remaining 4 particles occupying the third shell. We could, in terms of $\mathcal{U}(3)$ multiplets, designate the configuration as

$$\{0\}^2\{1\}^6\{2\}^4 \tag{13}$$

The spin and unitary $\mathcal{U}(3)$ multiplets can be determined by first evaluating the plethysm

$$\{1\}\{2\}\otimes\{1^4\} \tag{14}$$

for the direct product group $\mathcal{SU}(2)\times\mathcal{U}(3)$. This leads to a set of spin $S=2$ states arising from the $\mathcal{U}(3)$ plethysm $\{2\}\otimes\{1^4\}$, a set of $S=1$ states from the $\mathcal{U}(3)$ plethysm $\{2\}\otimes\{21^2\}$ and a set of $S=0$ states from the $\mathcal{U}(3)$ plethysm $\{2\}\otimes\{2^2\}$. The first two filled shells result in a single $S=0$ state transforming under $\mathcal{U}(3)$ as the $\{2^3\}$ and to obtain the final list of $\mathcal{U}(3)$ irreducible representations we must add the partition $\{2^3\}$ to those associated with each of the above plethysms to finally yield the spin S and $\mathcal{U}(3)$ multiplets given in Table 6.1.

$S=2$	$\{653\}$				
$S=1$	$\{83^2\}$	$+\ \{752\}$	$+\ \{743\}$	$+\ \{653\}$	$+\ \{5^24\}$
$S=0$	$\{842\}$	$+\ \{743\}$	$+\ \{6^22\}$	$+\ \{64^2\}$	

Table 6.1 Spin and $\mathcal{U}(3)$ multiplets for the $\{0\}^2\{1\}^6\{2\}^4$ configuration.

7. Second to Lowest Energy Even Parity 12-particle States

The second to lowest energy even parity 12-particle states all involve $\mathcal{U}(3)$ multiplets labelled by partitions of weight 16. Five configurations arise:-

1. $\{0\}^2\{1\}^5\{2\}^4\{3\}^1$
2. $\{0\}^1\{1\}^6\{2\}^5$
3. $\{0\}^2\{1\}^4\{2\}^6$
4. $\{0\}^2\{1\}^6\{2\}^3\{4\}^1$
5. $\{0\}^2\{1\}^6\{2\}^2\{3\}^2$

Proceeding as before we can systematically determine the various possible spins S and their associated $\mathcal{U}(3)$ multiplets to give:-

$S = 1, 2^2, 3$	$\{952\}$	$+ \{943\}$	$+ \{862\}$	$+ 3\{853\}$	$+ \{84^2\}$
	$+ 2\{763\}$	$+ 3\{754\}$	$+ 2\{6^24\}$	$+ 2\{65^2\}$	
$S = 0, 1^2, 2$	$\{11\ 32\}$	$+ \{10\ 51\}$	$+ 3\{10\ 42\}$	$+ 3\{10\ 3^2\}$	$+ \{961\}$
	$+ 6\{952\}$	$+ 7\{943\}$	$+ \{871\}$	$+ 6\{862\}$	$+ 12\{853\}$
	$+ 5\{84^2\}$	$+ 3\{7^22\}$	$+ 9\{763\}$	$+ 10\{754\}$	$+ 4\{6^24\}$
	$+ 4\{65^2\}$				
$S = 0, 1$	$\{11\ 41\}$	$+ \{11\ 32\}$	$+ \{10\ 51\}$	$+ 4\{10\ 42\}$	$+ 2\{10\ 3^2\}$
	$+ 2\{961\}$	$+ 5\{952\}$	$+ 7\{943\}$	$+ \{871\}$	$+ 6\{862\}$
	$+ 8\{853\}$	$+ 6\{84^2\}$	$+ 2\{7^22\}$	$+ 7\{763\}$	$+ 6\{754\}$
	$+ 4\{6^24\}$	$+ \{65^2\}$			

Table 7.1 Spin and $\mathcal{U}(3)$ multiplets for the $\{0\}^2\{1\}^5\{2\}^4\{3\}^1$ configuration.

$S = 2, 3$	$\{6^24\}$				
$S = 1, 2$	$\{853\}$	$+ \{763\}$	$+ \{754\}$	$+ \{65^2\}$	
$S = 0, 1$	$\{943\}$	$+ \{862\}$	$+ \{853\}$	$+ \{84^2\}$	$+ \{763\}$
	$+ \{754\}$	$+ \{6^24\}$			

Table 7.2 Spin and $\mathcal{U}(3)$ multiplets for the $\{0\}^1\{1\}^6\{2\}^5$ configuration.

$S = 2, 3, 4$	$\{65^2\}$				
$S = 3$	$\{6^24\}$				
$S = 1, 2, 3$	$\{853\}$	$+ \{763\}$	$+ 2\{754\}$	$+ \{6^24\}$	$+ \{65^2\}$
$S = 2$	$\{862\}$	$+ \{853\}$	$+ \{84^2\}$	$+ 2\{763\}$	$+ 2\{754\}$
	$+ 2\{6^24\}$	$+ \{65^2\}$			
$S = 0, 1, 2$	$\{952\}$	$+ \{943\}$	$+ 2\{862\}$	$+ 3\{853\}$	$+ 2\{84^2\}$
	$+ \{7^22\}$	$+ 3\{763\}$	$+ 3\{754\}$	$+ 2\{6^24\}$	$+ \{65^2\}$
$S = 1$	$\{961\}$	$+ 2\{952\}$	$+ 2\{943\}$	$+ \{871\}$	$+ 3\{862\}$
	$+ 5\{853\}$	$+ 2\{84^2\}$	$+ 2\{7^22\}$	$+ 5\{763\}$	$+ 5\{754\}$
	$+ 2\{6^24\}$	$+ 2\{65^2\}$			
$S = 1$	$\{10\ 3^2\}$	$+ \{952\}$	$+ \{943\}$	$+ \{871\}$	$+ \{862\}$
	$+ 3\{853\}$	$+ \{7^22\}$	$+ 2\{763\}$	$+ 2\{754\}$	$+ \{65^2\}$
$S = 0$	$\{10\ 42\}$	$+ \{961\}$	$+ \{952\}$	$+ 2\{943\}$	$+ \{8^2\}$
	$+ \{871\}$	$+ 4\{862\}$	$+ 3\{853\}$	$+ 3\{84^2\}$	$+ 3\{763\}$
	$+ 2\{754\}$	$+ 3\{6^24\}$			

Table 7.3 Spin and $\mathcal{U}(3)$ multiplets for the $\{0\}^2\{1\}^4\{2\}^6$ configuration.

$S = 1, 2$	$\{10\ 3^2\}$	$+ \{952\}$	$+ \{943\}$	$+ 2\{853\}$
	$+ \{763\}$	$+ \{754\}$	$+ \{65^2\}$	
$S = 0, 1$	$\{11\ 32\}$	$+ 2\{10\ 42\}$	$+ \{10\ 3^2\}$	$+ 2\{952\}$
	$+ 3\{943\}$	$+ 2\{862\}$	$+ 3\{853\}$	$+ 2\{84^2\}$
	$+ \{7^22\}$	$+ 2\{763\}$	$+ 2\{754\}$	$+ \{6^24\}$

Table 7.4 Spin and $\mathcal{U}(3)$ multiplets for the $\{0\}^2\{1\}^6\{2\}^3\{4\}$ configuration.

$S=0,1,2$	$\{10\ 42\}$	$+\ \{10\ 3^2\}$	$+\ \{952\}$	$+\ 2\{943\}$
	$+\ 2\{862\}$	$+\ 3\{853\}$	$+\ \{84^2\}$	$+\ 2\{763\}$
	$+\ 2\{754\}$	$+\ \{6^24\}$	$+\ \{65^2\}$	
$S=0$	$\{12\ 2^2\}$	$+\ \{11\ 32\}$	$+\ 3\{10\ 42\}$	$+\ 2\{952\}$
	$+\ 2\{943\}$	$+\ 3\{862\}$	$+\ 2\{853\}$	$+\ 3\{84^2\}$
	$+\ 2\{763\}$	$+\ 2\{754\}$	$+\ 2\{6^24\}$	
$S=1$	$2\{11\ 32\}$	$+\ 2\{10\ 42\}$	$+\ 2\{10\ 3^2\}$	$+\ 5\{952\}$
	$+\ 4\{943\}$	$+\ 2\{862\}$	$+\ 6\{853\}$	$+\ 2\{84^2\}$
	$+\ 3\{7^22\}$	$+\ 4\{763\}$	$+\ 5\{754\}$	$+\ \{6^24\}$
	$+\ 2\{65^2\}$			

Table 7.5 Spin and $\mathcal{U}(3)$ multiplets for the $\{0\}^2\{1\}^6\{2\}^2\{3\}^2$ configuration.

In practice there is no difficulty in obtaining the corresponding odd parity 12-particle states. Of course the total number of possible states is infinite and to encompass these we must return to the non-compact groups.

8. Infinite Sets of Even Parity 12-particle States

The complete set of even parity 12-particle states span the infinite dimensional irreducible representation $\langle s;(0)\rangle$ of the non-compact group $\mathcal{S}p(72,R)$. To obtain a description of the states we need to study the decomposition of the irreducible representation $\langle s;(0)\rangle$ as we move through a series of subgroups as portrayed in Fig.1. Any such decomposition involves an infinite set of subgroup irreducible representations and hence to consider manageable problems we need to introduce a cutoff. For simplicity let us consider the restriction $\mathcal{S}p(72,R)\rightarrow\mathcal{S}p(6,R)\times\mathcal{O}(12)$ and furthermore limit our attention to irreducible representations whose labelling partitions (λ) are of weight $w_\lambda\leq 16$. We readily find the decomposition as given in Table 8.1.

$\langle s;(0)\rangle$	$\langle 6;(16)\rangle[16]$	$+\ \langle 6;(15\,1)\rangle[15\,1]$	$+\ \langle 6;(14\,2)\rangle[14\,2]$
	$+\ \langle 6;(14\,1^2)\rangle[14\,1^2]$	$+\ \langle 6;(14)\rangle[14]$	$+\ \langle 6;(13\,3)\rangle[13\,3]$
	$+\ \langle 6;(13\,21)\rangle[13\,21]$	$+\ \langle 6;(13\,1)\rangle[13\,1]$	$+\ \langle 6;(12\,4)\rangle[12\,4]$
	$+\ \langle 6;(12\,31)\rangle[12\,31]$	$+\ \langle 6;(12\,2^2)\rangle[12\,2^2]$	$+\ \langle 6;(12\,2)\rangle[12\,2]$
	$+\ \langle 6;(12\,1^2)\rangle[12\,1^2]$	$+\ \langle 6;(12)\rangle[12]$	$+\ \langle 6;(11\,5)\rangle[11\,5]$
	$+\ \langle 6;(11\,41)\rangle[11\,41]$	$+\ \langle 6;(11\,32)\rangle[11\,32]$	$+\ \langle 6;(11\,3)\rangle[11\,3]$
	$+\ \langle 6;(11\,21)\rangle[11\,21]$	$+\ \langle 6;(11\,1)\rangle[11\,1]$	$+\ \langle 6;(10\,6)\rangle[10\,6]$
	$+\ \langle 6;(10\,51)\rangle[10\,51]$	$+\ \langle 6;(10\,42)\rangle[10\,42]$	$+\ \langle 6;(10\,4)\rangle[10\,4]$
	$+\ \langle 6;(10\,3^2)\rangle[10\,3^2]$	$+\ \langle 6;(10\,31)\rangle[10\,31]$	$+\ \langle 6;(10\,2^2)\rangle[10\,2^2]$
	$+\ \langle 6;(10\,2)\rangle[10\,2]$	$+\ \langle 6;(10\,1^2)\rangle[10\,1^2]$	$+\ \langle 6;(10)\rangle[10]$
	$+\ \langle 6;(97)\rangle[97]$	$+\ \langle 6;(961)\rangle[961]$	$+\ \langle 6;(952)\rangle[952]$
	$+\ \langle 6;(95)\rangle[95]$	$+\ \langle 6;(943)\rangle[943]$	$+\ \langle 6;(941)\rangle[941]$
	$+\ \langle 6;(932)\rangle[932]$	$+\ \langle 6;(93)\rangle[93]$	$+\ \langle 6;(921)\rangle[921]$
	$+\ \langle 6;(91)\rangle[91]$	$+\ \langle 6;(8^2)\rangle[8^2]$	$+\ \langle 6;(871)\rangle[871]$
	$+\ \langle 6;(862)\rangle[862]$	$+\ \langle 6;(86)\rangle[86]$	$+\ \langle 6;(853)\rangle[853]$
	$+\ \langle 6;(851)\rangle[851]$	$+\ \langle 6;(84^2)\rangle[84^2]$	$+\ \langle 6;(842)\rangle[842]$
	$+\ \langle 6;(84)\rangle[84]$	$+\ \langle 6;(83^2)\rangle[83^2]$	$+\ \langle 6;(831)\rangle[831]$
	$+\ \langle 6;(82^2)\rangle[82^2]$	$+\ \langle 6;(82)\rangle[82]$	$+\ \langle 6;(81^2)\rangle[81^2]$
	$+\ \langle 6;(8)\rangle[8]$	$+\ \langle 6;(7^22)\rangle[7^22]$	$+\ \langle 6;(7^2)\rangle[7^2]$
	$+\ \langle 6;(763)\rangle[763]$	$+\ \langle 6;(761)\rangle[761]$	$+\ \langle 6;(754)\rangle[754]$
	$+\ \langle 6;(752)\rangle[752]$	$+\ \langle 6;(75)\rangle[75]$	$+\ \langle 6;(743)\rangle[743]$
	$+\ \langle 6;(741)\rangle[741]$	$+\ \langle 6;(732)\rangle[732]$	$+\ \langle 6;(73)\rangle[73]$
	$+\ \langle 6;(721)\rangle[721]$	$+\ \langle 6;(71)\rangle[71]$	$+\ \langle 6;(6^24)\rangle[6^24]$
	$+\ \langle 6;(6^22)\rangle[6^22]$	$+\ \langle 6;(6^2)\rangle[6^2]$	$+\ \langle 6;(65^2)\rangle[65^2]$
	$+\ \langle 6;(653)\rangle[653]$	$+\ \langle 6;(651)\rangle[651]$	$+\ \langle 6;(64^2)\rangle[64^2]$
	$+\ \langle 6;(642)\rangle[642]$	$+\ \langle 6;(64)\rangle[64]$	$+\ \langle 6;(63^2)\rangle[63^2]$
	$+\ \langle 6;(631)\rangle[631]$	$+\ \langle 6;(62^2)\rangle[62^2]$	$+\ \langle 6;(62)\rangle[62]$
	$+\ \langle 6;(61^2)\rangle[61^2]$	$+\ \langle 6;(6)\rangle[6]$	$+\ \langle 6;(5^24)\rangle[5^24]$
	$+\ \langle 6;(5^22)\rangle[5^22]$	$+\ \langle 6;(5^2)\rangle[5^2]$	$+\ \langle 6;(543)\rangle[543]$
	$+\ \langle 6;(541)\rangle[541]$	$+\ \langle 6;(532)\rangle[532]$	$+\ \langle 6;(53)\rangle[53]$
	$+\ \langle 6;(521)\rangle[521]$	$+\ \langle 6;(51)\rangle[51]$	$+\ \langle 6;(4^3)\rangle[4^3]$
	$+\ \langle 6;(4^22)\rangle[4^22]$	$+\ \langle 6;(4^2)\rangle[4^2]$	$+\ \langle 6;(43^2)\rangle[43^2]$
	$+\ \langle 6;(431)\rangle[431]$	$+\ \langle 6;(42^2)\rangle[42^2]$	$+\ \langle 6;(42)\rangle[42]$
	$+\ \langle 6;(41^2)\rangle[41^2]$	$+\ \langle 6;(4)\rangle[4]$	$+\ \langle 6;(3^22)\rangle[3^22]$
	$+\ \langle 6;(3^2)\rangle[3^2]$	$+\ \langle 6;(321)\rangle[321]$	$+\ \langle 6;(31)\rangle[31]$
	$+\ \langle 6;(2^3)\rangle[2^3]$	$+\ \langle 6;(2^2)\rangle[2^2]$	$+\ \langle 6;(21^2)\rangle[21^2]$
	$+\ \langle 6;(2)\rangle[2]$	$+\ \langle 6;(1^2)\rangle[1^2]$	$+\ \langle 6;(0)\rangle[0]$

Table 8.1 Decomposition of the irreducible representation $\langle s;(0)\rangle$ of $\mathcal{S}p(72,R)$ under the restriction $\mathcal{S}p(72,R) \to \mathcal{S}p(6,R) \times \mathcal{O}(12)$ (to weight 16).

This is already a considerable list of irreducible representations. The list can be substantially reduced by noting that no partition (λ) of weight $w_\lambda \leq 13$ can yield a Pauli allowed spin state and hence all those members of the list may be removed. Under the restriction $\mathcal{S}p(6,R) \to \mathcal{U}(3)$ for an irreducible representation $\langle 6;(\lambda)\rangle$ the lowest weight $\mathcal{U}(3)$ irreducible representation is necessarily $\{\lambda\}$ and as seen from Sec. 4 partitions into fewer than three parts cannot lead to a Pauli allowed spin state and hence all irreducible representations associated with partitions into fewer than three parts may also be discarded leaving us with the much shorter list shown in Table 8.2.

$\langle s;(0)\rangle$	$\langle 6;(14\,1^2)\rangle[14\,1^2]$	$+\langle 6;(13\,21)\rangle[13\,21]$	$+\langle 6;(12\,31)\rangle[12\,31]$
	$+\langle 6;(12\,2^2)\rangle[12\,2^2]$	$+\langle 6;(12\,1^2)\rangle[12\,1^2]$	$+\langle 6;(11\,41)\rangle[11\,41]$
	$+\langle 6;(11\,32)\rangle[11\,32]$	$+\langle 6;(11\,21)\rangle[11\,21]$	$+\langle 6;(10\,51)\rangle[10\,51]$
	$+\langle 6;(10\,42)\rangle[10\,42]$	$+\langle 6;(10\,3^2)\rangle[10\,3^2]$	$+\langle 6;(10\,31)\rangle[10\,31]$
	$+\langle 6;(10\,2^2)\rangle[10\,2^2]$	$+\langle 6;(961)\rangle[961]$	$+\langle 6;(952)\rangle[952]$
	$+\langle 6;(943)\rangle[943]$	$+\langle 6;(941)\rangle[941]$	$+\langle 6;(932)\rangle[932]$
	$+\langle 6;(871)\rangle[871]$	$+\langle 6;(862)\rangle[862]$	$+\langle 6;(853)\rangle[853]$
	$+\langle 6;(851)\rangle[851]$	$+\langle 6;(84^2)\rangle[84^2]$	$+\langle 6;(842)\rangle[842]$
	$+\langle 6;(83^2)\rangle[83^2]$	$+\langle 6;(7^22)\rangle[7^22]$	$+\langle 6;(763)\rangle[763]$
	$+\langle 6;(761)\rangle[761]$	$+\langle 6;(754)\rangle[754]$	$+\langle 6;(752)\rangle[752]$
	$+\langle 6;(743)\rangle[743]$	$+\langle 6;(6^24)\rangle[6^24]$	$+\langle 6;(6^22)\rangle[6^22]$
	$+\langle 6;(65^2)\rangle[65^2]$	$+\langle 6;(653)\rangle[653]$	$+\langle 6;(64^2)\rangle[64^2]$
	$+\langle 6;(5^24)\rangle[5^24]$		

Table 8.2 As for Table 8.1 but with terms of weight < 14 and length < 3 removed.

The $\mathcal{O}(12)$ irreducible representations are all finite dimensional whereas those of $\mathcal{S}p(6,R)$ are all of infinite dimension. The reductions $\mathcal{S}p(6,R) \to \mathcal{U}(3)$ and $\mathcal{O}(12) \to \mathcal{S}(12)$ tell us the $\mathcal{U}(3)$ and spin contents respectively.

9. Spin Content of the 12-particle States

The spin content of the states associated with a given irreducible representation $[\lambda]$ of $\mathcal{O}(n)$ is determined by its decomposition under $\mathcal{O}(n) \to \mathcal{S}(n)$ and seeking out those irreducible representations of $\mathcal{S}(n)$ that are of the form $\{2^r\,1^s\}$ where $2r+s=n$ and the associated spin is $S=\frac{s}{2}$. These decompositions may be determined systematically[7,8,10,11]. Typically we obtain the spin states shown in Table 9.1 for several $\mathcal{O}(12)$ irreducible representations.

$S=$	0	1	2	3	4
$[5^2\,4]$		1			
$[64^2]$	1				
$[653]$		1	1		
$[6^2\,2]$	1				
$[743]$	1	1			
$[752]$		1			
$[83^2]$		1			
$[842]$	1				
$[65^2]$	7	19	14	4	1
$[6^2\,4]$	16	22	14	5	
$[754]$	26	49	26	5	
$[763]$	27	46	22	3	
$[7^2\,2]$	7	15	4		
$[84^2]$	20	26	11	1	
$[853]$	33	59	28	4	
$[862]$	24	31	13	1	
$[871]$	3	5	1		
$[943]$	23	35	13	1	
$[952]$	17	30	11	1	
$[961]$	4	5	1		
$[10\,3^2]$	7	13	5		
$[10\,42]$	13	15	4		
$[10\,51]$	2	3	1		
$[11\,41]$	4	6	1		
$[12\,2^2]$	1				

Table 9.1 Spin contents of some relevant $\mathcal{O}$(12 irreducible representations.

Note that in going from partitions of weight 14 to 16 the number of possible spin states rapidly increasing in a manner not unlike the Wigner type distribution that arises in the plotting of the distribution of the spacings of consecutive eigenvalues of large random matrices. A similar effect has been observed in other group-subgroup decompositions and merits more study[12-14].

An important, and as yet incompletely solved,problem is to be able to predict those irreducible representations of $\mathcal{O}(n)$ that cannot yield irreducible representations of $\mathcal{S}(n)$ of the form $\{2^r1^s\}$ without requiring an explicit decomposition. A further problem is to develop a method that will directly yield the multiplicity of a given irreducible representation of the form $\{2^r1^s\}$ without requiring a complete decomposition under $\mathcal{O}(n) \to \mathcal{S}(n)$. A key to the evaluation of such decompositions is the evaluation of so-called reduced plethysms[10,11] of the form $\langle 1\rangle \otimes \{\lambda\}$. Hints at a solution come from the observation that if (λ) is a one part partition, say (k), then increasing k in steps of unity results for a certain value of k the multiplicity coefficient of say $\langle\mu_1, \mu_2, \ldots\rangle$ and $\langle\mu_1 + 1, \mu_2, \ldots\rangle$ being equal. Thereafter the multiplicity coefficients of $\langle\mu_1 + x, \mu_2, \ldots\rangle$ are independent of x and are said to be *stabilised.* The coefficients up to the stabilisation point often form identifiable integer sequences[15]. Scharf and Thibon [16] have used such considerations to recently derive a generating function for the multiplicity coefficients that arise in $\langle 1\rangle \otimes \{\lambda\}$.

10. $\mathcal{U}(3)$ Content of the 12-particle States

The $\mathcal{U}(3)$ content of the 12-particle states comes from the decomposition of the irreducible representations of $\mathcal{S}p(6, R)$ under the group reduction $\mathcal{S}p(6, R) \to \mathcal{U}(3)$ [2,3]. Some relevant decompositions are given below for $\mathcal{U}(3)$ irreducible representations, truncated at weight 18 are given in Table 10.1.

$\langle 6;(5^2\ 4)\rangle$	$\{954\}$	$+\ \{85^2\}$	$+\ \{7^2 4\}$	$+\ \{765\}$	$+\ \{754\}$
	$+\ \{65^2\}$	$+\ \{5^2 4\}$			
$\langle 6;(64^2)\rangle$	$\{10\ 4^2\}$	$+\ \{954\}$	$+\ 2\{864\}$	$+\ \{84^2\}$	$+\ \{765\}$
	$+\ \{754\}$	$+\ \{6^3\}$	$+\ \{6^2 4\}$	$+\ \{64^2\}$	
$\langle 6;(653)\rangle$	$\{10\ 53\}$	$+\ \{963\}$	$+\ \{954\}$	$+\ \{873\}$	$+\ 2\{864\}$
	$+\ 2\{85^2\}$	$+\ \{853\}$	$+\ \{7^2 4\}$	$+\ 2\{765\}$	$+\ \{763\}$
	$+\ \{754\}$	$+\ \{6^2 4\}$	$+\ \{65^2\}$	$+\ \{653\}$	
$\langle 6;(6^2\ 2)\rangle$	$\{10\ 62\}$	$+\ \{963\}$	$+\ \{8^2 2\}$	$+\ \{873\}$	$+\ 2\{864\}$
	$+\ \{862\}$	$+\ \{765\}$	$+\ \{763\}$	$+\ \{6^3\}$	$+\ \{6^2 4\}$
	$+\ \{6^2 2\}$				
$\langle 6;(743)\rangle$	$\{11\ 43\}$	$+\ \{10\ 53\}$	$+\ \{10\ 4^2\}$	$+\ 2\{963\}$	$+\ 2\{954\}$
	$+\ \{943\}$	$+\ \{873\}$	$+\ 2\{864\}$	$+\ \{85^2\}$	$+\ \{853\}$
	$+\ \{84^2\}$	$+\ \{7^2 4\}$	$+\ \{765\}$	$+\ \{763\}$	$+\ \{754\}$
	$+\ \{743\}$				
$\langle 6;(752)\rangle$	$\{11\ 52\}$	$+\ \{10\ 62\}$	$+\ \{10\ 53\}$	$+\ 2\{972\}$	$+\ 2\{963\}$
	$+\ 2\{954\}$	$+\ \{952\}$	$+\ 2\{873\}$	$+\ 2\{864\}$	$+\ \{862\}$
	$+\ \{85^2\}$	$+\ \{853\}$	$+\ 2\{7^2 4\}$	$+\ \{7^2 2\}$	$+\ \{765\}$
	$+\ \{763\}$	$+\ \{754\}$	$+\ \{752\}$		
$\langle 6;(83^2)\rangle$	$\{12\ 3^2\}$	$+\ \{11\ 43\}$	$+\ 2\{10\ 53\}$	$+\ \{10\ 3^2\}$	$+\ \{963\}$
	$+\ \{954\}$	$+\ \{943\}$	$+\ \{873\}$	$+\ \{85^2\}$	$+\ \{853\}$
	$+\ \{83^2\}$				
$\langle 6;(842)\rangle$	$\{12\ 42\}$	$+\ \{11\ 52\}$	$+\ \{11\ 43\}$	$+\ 2\{10\ 62\}$	$+\ 2\{10\ 53\}$
	$+\ 2\{10\ 4^2\}$	$+\ \{10\ 42\}$	$+\ \{972\}$	$+\ 2\{963\}$	$+\ 2\{954\}$
	$+\ \{952\}$	$+\ \{943\}$	$+\ \{8^2 2\}$	$+\ \{873\}$	$+\ 2\{864\}$
	$+\ \{862\}$	$+\ \{853\}$	$+\ \{84^2\}$	$+\ \{842\}$	
$\langle 6;(65^2)\rangle$	$\{85^2\}$	$+\ \{765\}$	$+\ \{65^2\}$		
$\langle 6;(6^2\ 4)\rangle$	$\{864\}$	$+\ \{765\}$	$+\ \{6^3\}$	$+\ \{6^2 4\}$	
$\langle 6;(754)\rangle$	$\{954\}$	$+\ \{864\}$	$+\ \{85^2\}$	$+\ \{7^2 4\}$	$+\ \{765\}$
	$+\ \{754\}$				
$\langle 6;(763)\rangle$	$\{963\}$	$+\ \{873\}$	$+\ \{864\}$	$+\ \{7^2 4\}$	$+\ \{765\}$
	$+\ \{763\}$				
$\langle 6;(7^2\ 2)\rangle$	$\{972\}$	$+\ \{873\}$	$+\ \{7^2 4\}$	$+\ \{7^2 2\}$	
$\langle 6;(84^2)\rangle$	$\{10\ 4^2\}$	$+\ \{954\}$	$+\ \{864\}$	$+\ \{84^2\}$	

Table 10.1 Some $\mathcal{S}p(6,R) \to \mathcal{U}(3)$ decompositions (to weight 18).

11. Orbital Angular Momentum of 12-particle States

The orbital angular momentum L of the 12-particle states follows from the decomposition of the $\mathcal{U}(3)$ irreducible representations under the restriction $\mathcal{U}(3) \to \mathcal{SO}(3)$. Considerable simplification arises by recognising that the irreducible representations of $\mathcal{U}(3)$ are irreducible under the restriction $\mathcal{U}(3) \to \mathcal{SU}(3)$ and for $\mathcal{SU}(3)$ the three part labelling partitions are equivalent to irreducible representations involving partitions into fewer than three parts, indeed

$$\{\lambda_1, \lambda_2, \lambda_3\} \equiv \{\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, 0\} \tag{15}$$

Thus the decomposition of the irreducible representation $\{5^2\ 4\}$ of $\mathcal{U}(3)$ is the same as that of the $\mathcal{SU}(3)$ irreducible representation $\{1\}$. Likewise the decompositions of the $\mathcal{U}(3)$ irreducible representations $\{65^2\}$ and $\{5^2\ 4\}$ are identical with respect to reduction to the subgroup $\mathcal{SO}(3)$. Likewise irreducible representations of $\mathcal{SU}(3)$ that are contragredient to one another i.e.,

$$\{\lambda_1, \lambda_2, \lambda_3\} \quad \text{and} \quad \{\lambda_1 - \lambda_3, \lambda_1 - \lambda_2, 0\} \tag{16}$$

have equivalent decompositions with respect to reduction to the subgroup $\mathcal{SO}(3)$. Some relevant $\mathcal{SU}(3) \to \mathcal{SO}(3)$ decompositions are given in Table 11.1.

$L=$	0	1	2	3	4	5	6	7	8	9
$\{0\}$	1									
$\{1\}$		1								
$\{2\}$	1		1							
$\{21\}$		1	1							
$\{3\}$		1		1						
$\{31\}$		1	1	1						
$\{4\}$	1		1		1					
$\{41\}$		1	1	1	1					
$\{42\}$	1		2	1	1					
$\{5\}$		1		1		1				
$\{51\}$		1	1	1	1	1				
$\{52\}$		1	1	2	1	1				
$\{6\}$	1		1		1		1			
$\{61\}$		1	1	1	1	1	1			
$\{62\}$	1		2	1	2	1	1			
$\{63\}$		1	1	2	2	1	1			
$\{7\}$		1		1		1		1		
$\{71\}$		1	1	1	1	1	1	1		
$\{72\}$		1	1	2	1	2	1	1		
$\{73\}$		1	1	2	2	2	1	1		
$\{8\}$	1		1		1		1		1	
$\{81\}$		1	1	1	1	1	1	1	1	
$\{82\}$	1		2	1	2	1	2	1	1	
$\{83\}$		1	1	2	2	2	2	1	1	
$\{84\}$	1		2	1	3	2	2	1	1	
$\{93\}$		1	1	2	2	2	2	2	1	1

Table 11.1 Some relevant $\mathcal{SU}(3) \to \mathcal{SO}(3)$ decompositions.

12. Labelling of the Even Parity 12-particle States

In the preceding I have outlined how one can systematically label the even parity 12-particle states using the group chain

$$\mathcal{S}p(72,R) \supset \mathcal{S}p(6,R) \times \mathcal{O}(12) \supset \mathcal{U}(3) \times \mathcal{S}(12) \supset \mathcal{SO}(3) \times \mathcal{S}(12) \tag{17}$$

The last segment of the chain, $\mathcal{SO}(3) \times \mathcal{S}(12)$, yields the traditional orbital, L, and spin, S, quantum numbers. Specific 12-particle even parity states can be systematically designated by the sequence of irreducible representations associated with the sequence of groups $\mathcal{S}p(72,R)\, \mathcal{S}p(6,R)\, \mathcal{U}(3)\, \mathcal{S}(12)^S\, \mathcal{SO}(3)^L$ leading to the notation

$$|\langle s;(0)\rangle, \langle 6;(\lambda)\rangle \alpha\{\lambda\}\beta_S\gamma_L\ {}^{2S+1}L\rangle \tag{18}$$

where α, β_S, γ_L stand for any other numbers required to distinguish the various reduction multiplicities. Usually we will suppress the irreducible representation of $\mathcal{S}p(72,R)$. Using the customary spectroscopic notation for the orbital angular momentum L and the spin multiplicity $2S+1$ as a superscript we may designate the lowest energy even parity 12-particle states of the configuration $\{0\}^2\{1\}^6\{2\}^4$ as shown in Table 12.1.

$\lvert\langle 6;(5^2\,4)\rangle\{5^2\,4\}\ {}^3P\rangle$	$\lvert\langle 6;(64^2)\rangle\{64^2\}\ {}^1SD\rangle$	$\lvert\langle 6;(653)\rangle\{653\}\ {}^{5,3}PDF\rangle$
$\lvert\langle 6;(6^2\,2)\rangle\{6^2\,2\}\ {}^1SDG\rangle$	$\lvert\langle 6;(743)\rangle\{743\}\ {}^{3,1}PDFG\rangle$	$\lvert\langle 6;(752)\rangle\{752\}\ {}^3PDF_2GH\rangle$
$\lvert\langle 6;(83^2)\rangle\{83^2\}\ {}^3PFH\rangle$	$\lvert\langle 6;(842)\rangle\{842\}\rangle\ {}^1SD_2FG_2HI\rangle$	

Table 12.1 States of the 12-particle configuration $\{0\}^2\{1\}^6\{2\}^4$.

The entries in Table 12.1 may be compared with those given in Table 6.1. Each of the entries in Table 12.1 represent the lowest energy terms of an infinite tower of states with each floor of the tower increasing in energy by $2\hbar\omega$. Each floor of the tower involves several $\mathcal{U}(3)$ multiplets all labelled by partitions of the same weight. Thus in our example the first floor involves $\mathcal{U}(3)$ multiplets of weight 14, those of the second floor weight 16 and so on. Thus all the $\mathcal{U}(3)$ multiplets appearing in Table 6.1 occur on the ground floor of the tower while those in Tables 7.1 to 7.5 occur on the second floor etc. Each floor can involve various values of S and L. All the states associated with a given $\mathcal{S}p(6,R)$ irreducible representation $\langle 6;(\lambda)\rangle$ start from the floor involving partitions of weight w_λ and contribute just the $\mathcal{U}(3)$ multiplet $\{\lambda\}$ to that floor. Going to the next floor can result in the $\mathcal{S}p(6,R)$ irreducible representation contributing several different $\mathcal{U}(3)$ irreducible representations as can be seen from Table 10.1. These $\mathcal{U}(3)$ multiplets will all involve the same spin structure but may involve differing orbital angular momenta as may be seen in the examples shown in Table 12.2.

$\vert\langle 6;(5^2\,4)\rangle$	$\{5^2\,4\}\;^3P\rangle$	$\vert\langle 6;(64^2)\rangle$	$\{64^2\}\;^1SD\rangle$	$\vert\langle 6;(653)\rangle$	$\{653\}\;^{5,3}PDF\rangle$
	$\{65^2\}\;^3P\rangle$		$\{6^2\,4\}\;^1SD\rangle$		$\{65^2\}\;^{5,3}P\rangle$
	$\{754\}\;^3PDF\rangle$		$\{754\}\;^1PDF\rangle$		$\{6^2\,4\}\;^{5,3}SD\rangle$
			$\{84^2\}\;^1PDF\rangle$		$\{763\}\;^{5,3}PDFG\rangle$
					$\{853\}\;^{5,3}PDF_2GH\rangle$

Table 12.2 Examples of some weight 14 and 16 states.

The odd parity states appear on floors interspacing those of the even parity states. Again successive odd parity floors involve an increase in energy of $2\hbar\omega$. As we ascend the infinite tower we find they become increasingly densely packed with $\mathcal{U}(3)$ multiplets associated with various spins. Each $\mathcal{U}(3)$ multiplet $\{\lambda\}$ appearing on the first floor is the first member of an infinite column of $\mathcal{U}(3)$ multiplets arising from the $\mathcal{S}p(6,R)\to\mathcal{U}(3)$ reduction of the $\mathcal{S}p(6,R)$ irreducible representation $\langle 6;(\lambda)\rangle$. These columns penetrate each of the successive floors of the same parity. Thus on each floor there will be $\mathcal{U}(3)$ multiplets originating from irreducible representations of $\mathcal{S}p(6,R)$ that started from lower floors, other $\mathcal{U}(3)$ multiplets will be associated with $\mathcal{S}p(6,R)$ irreps that start from that floor (See Fig. 3). Not surprisingly we have infinite sets of infinite dimensional irreducible representations $\langle 6;(\lambda)\rangle$ each starting from the floor whose zero-order energy is $w_\lambda\hbar\omega$.

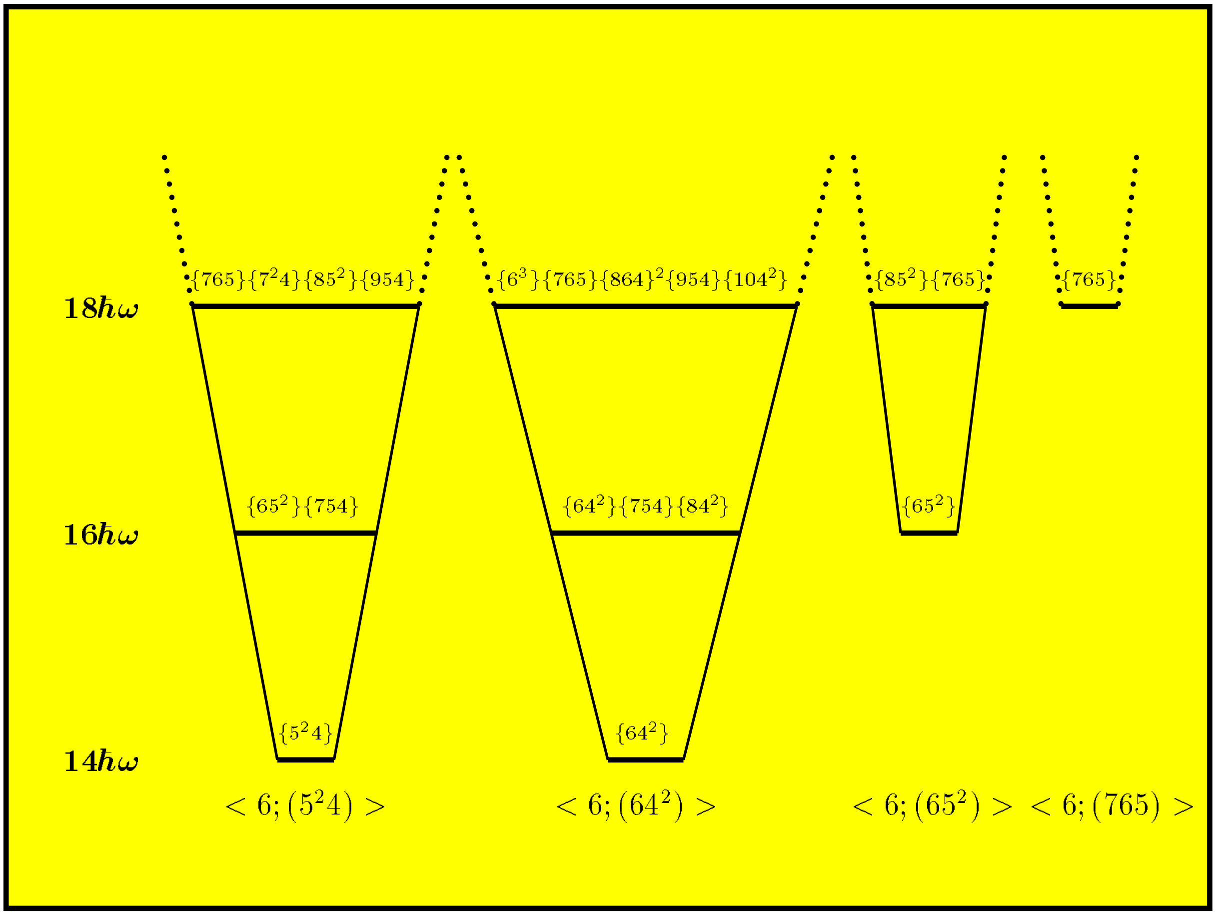

Fig. 3: Some of the infinite $\mathcal{S}p(6,R)$ multiplets for 12 particles showing the $\mathcal{U}(3)$ multiplets for the lowest three zero-order energy levels.

13. An Example

The self-consistency of the picture just outlined can be seen in the following example. We note from Tables 7.1 to 7.5 that the second floor contains the $\mathcal{U}(3)$ irreducible representation $\{853\}$. simply counting the entries in those tables shows that this $\mathcal{U}(3)$ irreducible representation occurs with the spins according to Table 13.1

$S=$	0	1	2	3
	35	63	29	4

Table 13.1 The number of times the $\mathcal{U}(3)$ irreducible representation $\{853\}$ occurs for the four allowed spin values.

At first sight it is tempting to associate all the above entries with the decomposition of the $\mathcal{O}(12)$ irreducible representation into those of $\mathcal{S}(12)$ and thence with the irreducible representation $\langle 6;(853)\rangle$ of $\mathcal{S}p(6,R)$. However, inspection of Table 9.1 shows that the spin content of the [853] irreducible representation of $\mathcal{O}(12)$ produces slightly fewer entries than in Table 14.1. Where have the extra irreducible representations $\{853\}$ of $\mathcal{U}(3)$ come from? The answer is clear if we inspect the entries in Table 10.1 and see that the weight 14 $\mathcal{S}p(6,R)$ irreducible representation can produce weight 16 irreducible representations of $\mathcal{U}(3)$. Thus the $\mathcal{S}p(6,R)$ irreducible representations

$$\langle 6;(653)\rangle,\quad \langle 6;(743)\rangle,\quad \langle 6;(752)\rangle,\quad \langle 6;(83^2)\rangle,\quad \langle 6;(842)\rangle \tag{19}$$

Inspection of Table 9.1 show that these give precisely the right number of spin multiplicities which when added to those coming from the $\langle 6;(853)\rangle\times[853]$ irreducible representation of $\mathcal{S}p(6,R)\times\mathcal{O}(12)$ reproduce the entries in Table 14.1 which demonstrates the full self-consistency of the non-compact group approach.

14. The Next Steps

In the preceding pages I have outlined how one can consistently establish a non-compact group description of the states of $n-$non-interacting fermions in an isotropic three-dimensional $\mathcal{HO}$. This part of the theory now appears to be fairly complete. The major remaining computational problem is associated with the rapid determination of the $\mathcal{O}(n)\to\mathcal{S}(n)$ decompositions. Significant progress has been made on this problem and further substantive progress can be expected.

The next step is to investigate model Hamiltonians constructed from polynomials in the group generators[]. A trivial example would be the introduction of a term proportional to $S(S+1)$ which would immediately separate terms according to their spins. If the term is positive then states of lowest spin would lie lowest as indeed the case for many-electron quantum dots[5]. The complete dynamical group $\mathcal{M}p(6n)$ has such a rich subgroup structure and its exploration has hardly begun. This is not surprising as the understanding of the properties of non-compact groups has been a comparatively recent development. In recent years there has been considerable progress in the systematic calculation of the matrix elements of non-compact group generators, a prequisite to undertaking detailed calculations[2,17].

While our discussion has been throughout devoted to three-dimensional systems there is no difficulty in increasing or decreasing the dimension of the system being considered.

Acknowledgements

This work has been assisted by Polish KBN Grant 18/p3/94/07. Much of the work reported herein was done while we were a guests of the Max Planck Astrophysik Institüt in Garching bei München. Part of this work forms the subject of a Master's thesis (KG).

All calculations were done using the C-package SCHUR*

* B. G. Wybourne, **SCHUR** is an interactive C package for calculating properties of Lie groups and symmetric functions. Distributed by: S. Christensen, P. O. Box 16175, Chapel Hill, NC 27516 USA. e-mail: steve@scm.vnet.net . A detailed description can be seen by WEB users at http://scm.vnet.net/Christensen.html

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