Aspects of the Exceptional Lie Groups

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Outline

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- John Stembridge's Problem
- An Example: The Symmetric Group
- The Exceptional Lie Groups
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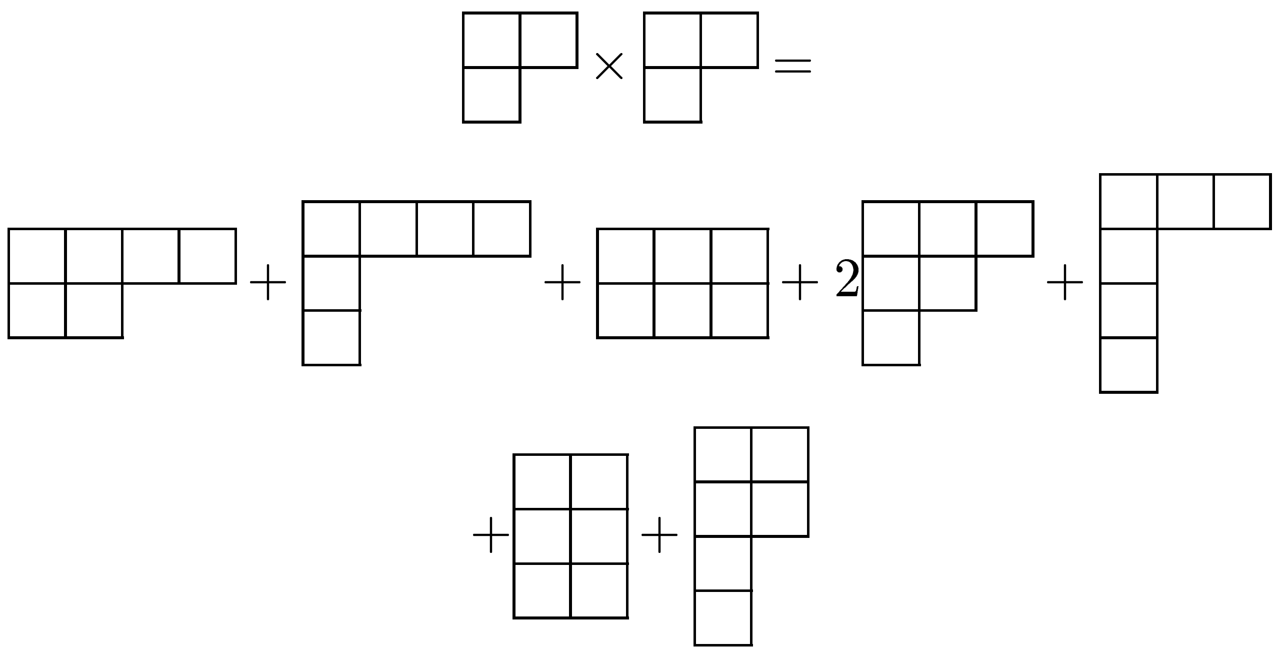
- *Whoever in the pursuit of science, seeks after immediate practical utility may rest assured that he seeks in vain*
  H. von Helmholtz (1862)

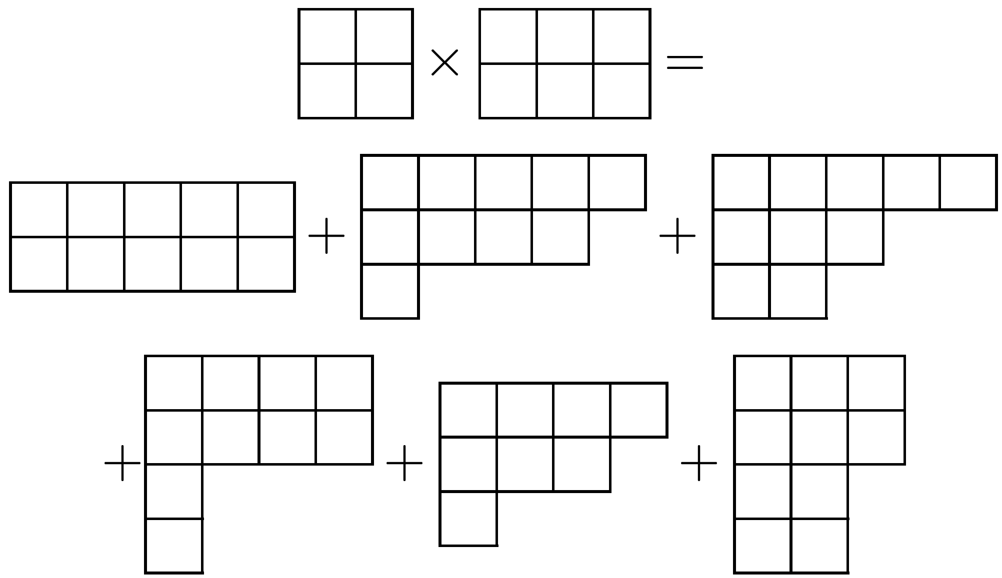
John Stembridge's Problem

- Multiplicity and the Littlewood-Richardson Rule
- Multiplying Young Tableaux



- What shapes yield multiplicity free products?

Example

- 

- Products of rectangles and squares are multiplicity free

I’ve got a little list

- SONG–KO-KO with CHORUS OF MEN.

  As some day it may happen that a victim must be found,
  I’ve got a little list–I’ve got a little list
  Of society offenders who might well be underground,
  And who never would be missed–who never would be missed!
  There’s the pestilential nuisances who write for autographs–
  All people who have flabby hands and irritating laughs–

  CHORUS. He’s got ’em on the list–he’s got ’em on the list;
  And they’ll none of ’em be missed–they’ll none of
  ’em be missed.
  (W. S. Gilbert: Mikado 14 March 1885, Savoy Theatre, London)

Example of the Symmetric group S_n

- $$\{n-1,1\}\circ\{n-1,1\} = \{n-2,2\}+\{n-2,1^2\}+\{n-1,1\}+\{n\}$$

$$\{n-1,1\}\circ\{n-a,a\} = \{n-a-1,a+1\}+\{n-a-1,a,1\}+\{n-a,a\}$$
$$+\{n-a,a-1,1\}+\{n-a+1,a-1\}$$

$$\{n-2,2\}\circ\{n-2,2\} = \{n-4,4\}+\{n-4,31\}+\{n-3,3\}+\{n-4,2^2\}$$
$$+2\{n-3,21\}+2\{n-2,2\}+\{n-3,1^3\}+\{n-2,1^2\}+\{n-1,1\}+\{n\}$$

The Exceptional Lie Groups

- Classification of complex semisimple Lie algebras completed by the 25 year old Elie Cartan in his thesis of 1894
- Four infinite series of classical Lie algebras A_k, B_k, C_k, D_k associated with the corresponding Lie groups $SU_{k+1}, SO_{2k+1}, Sp_{2k}, SO_{2k}$
- $k = 1, 2, \ldots$ is the *rank* of the Lie algebra
- Cartan found 5 exceptional Lie algebras of fixed ranks, G_2, F_4, E_6, E_7, E_8

Maximal Classical Subgroups

$$\begin{array}{ccc} G_2 & \supset & SU_3 \\ F_4 & \supset & SO_9 \\ E_6 & \supset & SU_2 \times SU_6 \\ E_7 & \supset & SU_8 \\ E_8 & \supset & SU_9 \end{array}$$

"When I use a word," Humpty Dumpty said in a rather scornful tone, *"it means just what I choose it to mean - neither more nor less"*
Lewis Carroll *Through the Looking Glass* (1872)

Partition labelling of irreps

Group	**Label**	**Constraints**
G_2	$(\lambda) = (\lambda_1, \lambda_2)$	$\lambda_1 \geq 2\lambda_2$
F_4	$(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	$\lambda_1 \geq \lambda_2 + \lambda_3 + \lambda_4$
	$(\Delta; \lambda) = (\lambda_1 + \frac{1}{2}, \lambda_2 + \frac{1}{2}, \lambda_3 + \frac{1}{2}, \lambda_4 + \frac{1}{2})$	$\lambda_1 > \lambda_2 + \lambda_3 + \lambda_4$
E_6	$(\lambda) = (\lambda_1 : \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$	$\lambda_1 \geq \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6$
E_7	$(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$	$\lambda_1 \geq \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7$
E_8	$(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)$	

$$\lambda_1 \geq 2\lambda_2 + 2\lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8$$

Fundamental Irreps of the Exceptional Lie Groups

G	ω_i	$((a))$	(λ)	$Dimension$
G_2	$\omega_1 = \theta$	01	(21)	14
	$\omega_2 = \omega$	01	(1)	7
F_4	$\omega_1 = \theta$	1000	(1^2)	52
	ω_2	1000	(21^2)	1274
	ω_3	0100	$(\Delta; 1)$	273
	$\omega_4 = \omega$	0010	(1)	26

Fundamental Irreps of the Exceptional Lie Groups

G	ω_i	$((a))$	(λ)	*Dimension*
E_6	$\omega_1 = \omega$	100000	$(1:1)$	27
	ω_2	01000	$(2:1^2)$	351
	ω_3	001000	$(3:1^3)$	2925
	ω_4	000100	$(2:1^4)$	351
	$\omega_5 = \bar{\omega}$	000010	$(1:1^5)$	27
	$\omega_6 = \theta$	000001	$(2:0)$	78

Fundamental Irreps of the Exceptional Lie Groups

G	ω_i	$((a))$	(λ)	*Dimension*
E_7	$\omega_1 = \theta$	1000000	(21^6)	133
	ω_2	0100000	(31^5)	8645
	ω_3	0010000	(41^4)	365750
	ω_4	0001000	(31^3)	27664
	ω_5	0000100	(21^2)	1539
	ω_6	0000010	(1^2)	56
	ω_7	0000001	(2)	912

Fundamental Irreps of the Exceptional Lie Groups

G	ω_i	$((a))$	(λ)	*Dimension*
E_8	$\omega_1 = \omega = \theta$	10000000	(21^7)	248
	ω_2	01000000	(31^6)	30380
	ω_3	00100000	(41^5)	2450240
	ω_4	00010000	(51^4)	146325270
	ω_5	00001000	(61^3)	6899079264
	ω_6	00000100	(41^2)	6696000
	ω_7	00000010	(21)	3875
	ω_8	00000001	(3)	147250

Maximal Tensor product multiplicities $c(\lambda;\omega_i,\omega_j)$ for E_8

	(21^7)	(31^6)	(41^5)	(51^4)	(61^3)	(41^2)	(21)	(3)
(21^7)	1	1	1	1	1	1	1	1
(31^6)	1	3	4	5	7	4	2	3
(41^5)	1	4	9	15	27	10	2	5
(51^4)	1	5	15	40	81	18	2	6
(61^3)	1	7	27	81	214	33	3	10
(41^2)	1	4	10	18	33	10	2	5
(21)	1	2	2	2	3	2	1	2
(3)	1	3	5	6	10	5	2	3

Some Explicit Tensor Products $(\theta) \times (n\omega_i)$ for E_8

$$(21^7) \times (2n, n) = (2n+2, n+1, 1^6) + (2n+1, n, 1^5) + (2n+1, n-1) + (2n, n, 1^6) + (2n, n) + (2n, n-1, 1)$$

$$(21^7) \times (2n, n^7) = (2n+2, (n+1)^7) + (2n+1, n^6, n-1) + (2n, n^7) + (2n, n, (n-1)^6) + (2n-1, (n-1)^6, n-2) + (2n-2, (n-1)^7)$$

$$(21^7) \times (3n) = (3n+2, 1^7) + (3n+1, 21^6) + (3n+1, 1^5) + (3n, 1^6) + (3n, 1^3) + (3n) + (3n-1, 1)$$

Expansions for some E_7 products

$$(m^2) \times (2n, n^6) =$$

$$\sum_{r=0}^{\min([m/2],n)} \sum_{p=0}^{\min(m-2r,n-r)} \sum_{q=0}^{\min(m-p-2r,n-p-r)}$$

$$(2n + m - q - 2p - 2r, m + n - 2q - p - 2r, n - q - p, (n - q - p - r)^4)$$

$$(m^2) \times (n^2) =$$

$$\sum_{p=0}^{m} \sum_{q=0}^{m-p} \sum_{r=0}^{q} (n + m - 2p, n + m - 2p - q, q, r^4)$$

The Complete Multiplicity free Kronecker Products

G_2	$(1) \times (\nu)$	$for\ all\ (\nu)$
	$(21) \times (n)$	
	$(21) \times (2n, n)$	
	$(2) \times (2n, n)$	
F_4	$(1) \times (\nu)$	$for\ all\ (\nu)\ with\ either$
		$\nu_1 = \nu_2 + \nu_3 + \nu_4\ or\ \nu_4 = 0$
		$or\ both$
	$(1^2) \times (\nu)$	$(\nu) = (n), (n^2), (2n, n), (3n, n^3), (\Delta; 3n + 1, n^3)$
	$(\Delta; 1) \times (n^2)$	
	$(2) \times (n^2)$	

The Complete Multiplicity free Kronecker Products

E_6 $\quad (1:1) \times (\nu) \quad for\ all\ (\nu)$

$$(2:0) \times (n:n), (n:n^5), (2n:0), (2n:n^2), (2n:n^4), (3n:n^3)$$

$$(m:m) \times (n:n), (n:n^5), (2n:0), (2n:n^2), (2n:n^4), (n:np^4)$$

$$(m:m^5) \times (n:n), (n:n^5), (2n:0), (2n:n^2), (2n:n^4), (n:np^4)$$

E_7 $\quad (1^2) \times (\nu) \quad for\ all\ (\nu)$

$$(21^6) \times (2n), (n^2), (2n, n^2), (2n, n^6), (3n, n^3), (3n, n^5), (4n, n^5)$$

$$(2) \times (2n), (n^2), 2n, n^6)$$

$$(21^2) \times (n^2), (2n, n^6)$$

$$(2^2) \times (2n, n^2)$$

$$(m^2) \times (n^2), (2n), (2n, n^6)$$

The Complete Multiplicity free Kronecker Products

$$E_8 \quad (21^7) \times (2n, n), (2n, n^7), (3n), (3n, n^6), (4n, n^2)$$
$$(4n, n^5), (5n, n^4), (6n, n^4)$$
$$(21) \times (2n, n), (2n, n^7)$$

- *One can measure the importance of a scientific work by the number of earlier publications rendered superfluous*
  David Hilbert

- *He is a rather good mathematician, but he will never be as good as Schottky*
  G. Frobenius, in a letter recommending the appointment of David Hilbert at Gottingen

Scaled Fundamental Irreps and Stable Products

- A *Scaled Fundamental Irrep* $n\omega_i$ has each part of the weight ω_i of the fundamental irrep ω_i multiplied by the positive integer n
- Consider the tensor product of a pair of scaled fundamentals

$$(m\omega_i) \times (n\omega_j) = \sum_{\nu} g^{\nu}_{m\omega_i, n\omega_j} (\nu)$$

. The product is *stable* if

$$n \geq m\omega_{i,1}$$

and

$$(m\omega_i) \times (n + k\omega_j) = \sum_{\nu} g^{\nu}_{m\omega_i, n\omega_j} (\nu + k\omega_j)$$

.

Example of Stable Products

Consider the group G_2 and the scaled products $(m) \times (2n, n)$

$$
\begin{aligned}
&(1) \times (21) && = (31) + (2) + (1) && m = 1, n = 1 \\
&(2) \times (21) && = (41) + (31) + (3) + (21) + (2) + (1) && m = 2, n = 1 \\
&(2) \times (42) && = (62) + (52) + (51) + (42) + (41) + (4) && \\
& && +(31) + (3) + (2) && m = 2, n = 2 \\
&(2) \times (63) && = 83) + (73) + (72) + (63) + (62) + (61) && \\
& && +(52) + (51) + (41) && m = 2, n = 3
\end{aligned}
$$

Conclusions and Acknowledgements

- All possible multiplicity free tensor products have been identified for the complete set of exceptional Lie groups
- Explicit expansions for all of the multiplicity free tensor products have been obtained
- The problem is complete and closed
- A similar analysis can be given for the Classical Lie Groups
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- The work reported here is joint with R C King (University of Southampton) and is part of our collaborative research