$E_8 \rightarrow F_4 \times G_2$ **Branching Rules**

$(0) \rightarrow$ $(0)(0)$

$(21^7) \rightarrow$ $(1^2)(0)$ $+ (1)(1)$ $+ (0)(21)$

$(21) \rightarrow$

$(2)(0)$	$+ (s;1)(1)$	$+ (1^2)(21)$	$+ (1)(2)$
$+ (1)(1)$	$+ (0)(2)$	$+ (0)(0)$	

$(31^6) \rightarrow$

$(21^2)(0)$	$+ (21)(1)$	$+ (2)(21)$	$+ (2)(1)$
$+ (s;1)(2)$	$+ (s;1)(1)$	$+ (s;1)(0)$	$+ (1^2)(21)$
$+ (1^2)(2)$	$+ (1^2)(0)$	$+ (1)(31)$	$+ (1)(21)$
$+ (1)(2)$	$+ 2(1)(1)$	$+ (0)(3)$	$+ (0)(21)$
$+ (0)(1)$			

$(3) \rightarrow$

$(s;2)(1)$	$+ (s;2)(0)$	$+ (21^2)(21)$	$+ (21)(2)$
$+ (21)(1)$	$+ (2)(21)$	$+ (2)(2)$	$+ (2)(1)$
$+ (2)(0)$	$+ (s;1)(31)$	$+ (s;1)(21)$	$+ (s;1)(2)$
$+ 2(s;1)(1)$	$+ (1^2)(3)$	$+ (1^2)(21)$	$+ (1^2)(2)$
$+ (1^2)(1)$	$+ (1^2)(0)$	$+ (1)(31)$	$+ (1)(3)$
$+ (1)(21)$	$+ 2(1)(2)$	$+ 2(1)(1)$	$+ (1)(0)$
$+ (0)(31)$	$+ (0)(21)$	$+ (0)(2)$	

$(427) \rightarrow$

$(2^2)(0)$	$+ (21)(1)$	$+ (2)(2)$	$+ (2)(0)$
$+ (s;1)(21)$	$+ (s;1)(1)$	$+ (1^2)(21)$	$+ (1^2)(1)$
$+ (1)(31)$	$+ (1)(2)$	$+ (1)(1)$	$+ (1)(0)$
$+ (0)(42)$	$+ (0)(2)$	$+ (0)(0)$	

$(41^5) \rightarrow$

$(31^3)(0)$	$+ (31^2)(1)$	$+ (31)(21)$	$+ (31)(1)$
$+ (3)(2)$	$+ (3)(1)$	$+ (3)(0)$	$+ (s;21)(2)$
$+ (s;21)(1)$	$+ (s;21)(0)$	$+ (s;2)(31)$	$+ 2(s;2)(21)$
$+ 2(s;2)(2)$	$+ 3(s;2)(1)$	$+ (s;2)(0)$	$+ (2^2)(2)$
$+ (2^2)(0)$	$+ (21^2)(3)$	$+ 2(21^2)(21)$	$+ (21^2)(2)$
$+ 2(21^2)(1)$	$+ (21^2)(0)$	$+ 2(21)(31)$	$+ (21)(3)$
$+ 2(21)(21)$	$+ 3(21)(2)$	$+ 4(21)(1)$	$+ (21)(0)$
$+ (2)(42)$	$+ 2(2)(31)$	$+ (2)(3)$	$+ 2(2)(21)$
$+ 5(2)(2)$	$+ 3(2)(1)$	$+ 3(2)(0)$	$+ (s;1)(41)$
$+ 3(s;1)(31)$	$+ 2(s;1)(3)$	$+ 4(s;1)(21)$	$+ 4(s;1)(2)$
$+ 6(s;1)(1)$	$+ (s;1)(0)$	$+ (1^2)(41)$	$+ 2(1^2)(31)$
$+ 2(1^2)(3)$	$+ 4(1^2)(21)$	$+ 2(1^2)(2)$	$+ 3(1^2)(1)$
$+ (1)(42)$	$+ (1)(41)$	$+ (1)(4)$	$+ 4(1)(31)$
$+ 2(1)(3)$	$+ 3(1)(21)$	$+ 6(1)(2)$	$+ 4(1)(1)$
$+ 2(1)(0)$	$+ (0)(42)$	$+ (0)(4)$	$+ (0)(31)$
$+ (0)(3)$	$+ 3(0)(2)$	$+ (0)(1)$	$+ 2(0)(0)$

$$421^6) \to$$

$$
\begin{aligned}
&(31)(0) && + (3)(1) && + (s;21)(1) && + (s;2)(21) \\
&+ (s;2)(2) && + 2(s;2)(1) && + (s;2)(0) && + (2^2)(21) \\
&+ (21^2)(21) && + (21^2)(2) && + (21^2)(1) && + (21^2)(0) \\
&+ (21)(31) && + (21)(21) && + 2(21)(2) && + 3(21)(1) \\
&+ (21)(0) && + (2)(31) && + (2)(3) && + 3(2)(21) \\
&+ 2(2)(2) && + 3(2)(1) && + (2)(0) && + 2(s;1)(31) \\
&+ (s;1)(3) && + 2(s;1)(21) && + 4(s;1)(2) && + 4(s;1)(1) \\
&+ 2(s;1)(0) && + (1^2)(42) && + (1^2)(31) && + (1^2)(3) \\
&+ 2(1^2)(21) && + 3(1^2)(2) && + 2(1^2)(1) && + 2(1^2)(0) \\
&+ (1)(41) && + 3(1)(31) && + 2(1)(3) && + 3(1)(21) \\
&+ 4(1)(2) && + 5(1)(1) && + (1)(0) && + (0)(41) \\
&+ (0)(31) && + (0)(3) && + 2(0)(21) && + (0)(2) \\
&+ 2(0)(1)
\end{aligned}
$$

$$(41^2) \to$$

$$
\begin{aligned}
&(s;3)(1) && + (s;3)(0) && + (31^3)(21) && + (31^3)(1) \\
&+ (31^2)(2) && + (31^2)(1) && + (31^2)(0) && + (31)(21) \\
&+ (31)(2) && + (31)(1) && + (31)(0) && + (3)(21) \\
&+ (3)(2) && + 2(3)(1) && + (s;21)(31) && + (s;21)(21) \\
&+ (s;21)(2) && + 2(s;21)(1) && + 2(s;2)(31) && + (s;2)(3) \\
&+ 3(s;2)(21) && + 4(s;2)(2) && + 5(s;2)(1) && + 2(s;2)(0) \\
&+ (2^2)(3) && + (2^2)(21) && + (2^2)(1) && + (21^2)(42) \\
&+ (21^2)(31) && + (21^2)(3) && + 2(21^2)(21) && + 3(21^2)(2) \\
&+ 2(21^2)(1) && + 2(21^2)(0) && + (21)(41) && + 3(21)(31) \\
&+ 2(21)(3) && + 3(21)(21) && + 5(21)(2) && + 5(21)(1) \\
&+ 2(21)(0) && + (2)(41) && + 3(2)(31) && + 3(2)(3) \\
&+ 5(2)(21) && + 4(2)(2) && + 6(2)(1) && + (2)(0) \\
&+ (s;1)(42) && + (s;1)(41) && + (s;1)(4) && + 5(s;1)(31) \\
&+ 3(s;1)(3) && + 4(s;1)(21) && + 8(s;1)(2) && + 6(s;1)(1) \\
&+ 3(s;1)(0) && + (1^2)(42) && + (1^2)(41) && + (1^2)(4) \\
&+ 3(1^2)(31) && + 2(1^2)(3) && + 3(1^2)(21) && + 5(1^2)(2) \\
&+ 3(1^2)(1) && + 2(1^2)(0) && + (1)(42) && + 2(1)(41) \\
&+ (1)(4) && + 5(1)(31) && + 4(1)(3) && + 5(1)(21) \\
&+ 6(1)(2) && + 6(1)(1) && + (1)(0) && + (0)(41) \\
&+ 2(0)(31) && + 2(0)(3) && + 2(0)(21) && + (0)(2) \\
&+ 2(0)(1)
\end{aligned}
$$

$$(42) \rightarrow$$

$(4)(0)$

$+ (31^2)(21)$

$+ (3)(2)$

$+ (s;21)(2)$

$+ (s;2)(3)$

$+ (s;2)(0)$

$+ (21^2)(31)$

$+ 2(21^2)(1)$

$+ 2(21)(21)$

$+ (2)(42)$

$+ (2)(21)$

$+ (s;1)(41)$

$+ 3(s;1)(2)$

$+ 2(1^2)(3)$

$+ (1)(42)$

$+ 2(1)(3)$

$+ 2(1)(0)$

$+ 3(0)(2)$

$+ (s;3)(1)$

$+ (31^2)(1)$

$+ (3)(1)$

$+ (s;21)(1)$

$+ 2(s;2)(21)$

$+ (2^2)(42)$

$+ (21^2)(3)$

$+ (21)(41)$

$+ 3(21)(2)$

$+ (2)(4)$

$+ 5(2)(2)$

$+ 3(s;1)(31)$

$+ 5(s;1)(1)$

$+ 3(1^2)(21)$

$+ (1)(41)$

$+ 2(1)(21)$

$+ (0)(42)$

$+ 2(0)(0)$

$+ (31^3)(2)$

$+ (31)(21)$

$+ (3)(0)$

$+ (s;21)(0)$

$+ 3(s;2)(2)$

$+ (2^2)(2)$

$+ 2(21^2)(21)$

$+ 2(21)(31)$

$+ 3(21)(1)$

$+ 2(2)(31)$

$+ 2(2)(1)$

$+ 3(s;1)(3)$

$+ (1^2)(41)$

$+ (1^2)(2)$

$+ (1)(4)$

$+ 5(1)(2)$

$+ (0)(4)$

$+ (31^3)(0)$

$+ (31)(1)$

$+ (s;21)(31)$

$+ (s;2)(31)$

$+ 3(s;2)(1)$

$+ (2^2)(0)$

$+ (21^2)(2)$

$+ (21)(3)$

$+ (21)(0)$

$+ (2)(3)$

$+ 3(2)(0)$

$+ 4(s;1)(21)$

$+ 2(1^2)(31)$

$+ 3(1^2)(1)$

$+ 3(1)(31)$

$+ 3(1)(1)$

$+ (0)(31)$

$$E_8 \to SO_{16} \text{ Branching Rules}$$

$(0) \to$ $[0]$

$(21^7) \to$ $[1^2]$ $+ [s; 0]_+$

$(21) \to$ $[2]$ $+ [s; 1]_-$ $+ [1^4]$

$(31^6) \to$ $[21^2]$ $+ [s; 1^2]_+$ $+ [s; 1]_-$ $+ [1^6]$ $+ [1^2]$

$(3) \to$ $[s; 2]_+$ $+ [21^4]$ $+ [21^2]$ $+ [2]$ $+ [s; 1^3]_-$
$+ [s; 1]_-$ $+ [1^8]_-$

$(42^7) \to$ $[2^2]$ $+ [s; 1^2]_+$ $+ [1^8]_+$ $+ [1^4]$ $+ [s; 0]_+$
$+ [0]$

$(41^5) \to$ $[31^3]$ $+ [s; 21^2]_+$ $+ [s; 21]_-$ $+ [s; 2]_+$ $+ [2^3]$
$+ [2^2 1^4]$ $+ [2^2]$ $+ [21^6]$ $+ [21^4]$ $+ [21^2]$
$+ [2]$ $+ [s; 1^5]_-$ $+ [s; 1^3]_-$ $+ [s; 1^2]_+$ $+ [s; 1]_-$
$+ [1^8]_-$ $+ [1^4]$

$(421^6) \to$ $[31]$ $+ [s; 21]_-$ $+ [s; 2]_+$ $+ [2^2 1^2]$ $+ [21^6]$
$+ [21^4]$ $+ [21^2]$ $+ [s; 1^4]_+$ $+ [s; 1^3]_-$ $+ [s; 1^2]_+$
$+ [s; 1]_-$ $+ [1^6]$ $+ [1^4]$ $+ [1^2]$ $+ [s; 0]_+$

$(41^2) \to$ $[s; 3]_-$ $+ [31^5]$ $+ [31^3]$ $+ [31]$ $+ [s; 21^3]_-$
$+ [s; 21^2]_+$ $+ [s; 21]_-$ $+ [s; 2]_+$ $+ [2^3 1^2]$ $+ [2^2 1^6]_-$
$+ [2^2 1^2]$ $+ [21^6]$ $+ [21^4]$ $+ [21^2]$ $+ [s; 1^5]_-$
$+ [s; 1^3]_-$ $+ [s; 1^2]_+$ $+ [s; 1]_-$ $+ [1^6]$ $+ [1^2]$

$(42) \to$ $[4]$ $+ [s; 3]_-$ $+ [31^7]_-$ $+ [31^3]$ $+ [s; 21^3]_-$
$+ [s; 21]_-$ $+ [2^4]$ $+ [2^2 1^4]$ $+ [2^2]$ $+ [21^4]$
$+ [s; 1^4]_+$ $+ [s; 1^2]_+$ $+ [1^8]_+$ $+ [1^4]$ $+ [s; 0]_+$
$+ [0]$

$$SO_{16} \rightarrow SO_9 \times SO7 \textbf{ Branching Rules}$$

$[s;0]_+ \rightarrow \qquad [s;0][s;0]$

$[s;1]_+ \rightarrow \qquad [s;1][s;0] \qquad\qquad + [s;0][s;1] \qquad\qquad + [s;0][s;0]$

$[s;1^2]_+ \rightarrow \qquad [s;1^2][s;0] \qquad\quad + [s;1][s;1] \qquad\qquad + [s;1][s;0] \qquad\qquad + [s;0][s;1^2]$
$\qquad\qquad\qquad\;\; + [s;0][s;1] \qquad\quad + [s;0][s;0]$

$[s;1^3]_+ \rightarrow \qquad [s;1^3][s;0] \qquad\quad + [s;1^2][s;1] \qquad\quad + [s;1^2][s;0] \qquad\quad + [s;1][s;1^2]$
$\qquad\qquad\qquad\;\; + [s;1][s;1] \qquad\quad + [s;1][s;0] \qquad\qquad + [s;0][s;1^3] \qquad\quad + [s;0][s;1^2]$
$\qquad\qquad\qquad\;\; + [s;0][s;1] \qquad\quad + [s;0][s;0]$

$[s;1^4]_+ \rightarrow \qquad [s;1^4][s;0] \qquad\quad + [s;1^3][s;1] \qquad\quad + [s;1^3][s;0] \qquad\quad + [s;1^2][s;1^2]$
$\qquad\qquad\qquad\;\; + [s;1^2][s;1] \qquad\quad + [s;1^2][s;0] \qquad\quad + [s;1][s;1^3] \qquad\quad + [s;1][s;1^2]$
$\qquad\qquad\qquad\;\; + [s;1][s;1] \qquad\quad + [s;1][s;0] \qquad\qquad + [s;0][s;1^3] \qquad\quad + [s;0][s;1^2]$
$\qquad\qquad\qquad\;\; + [s;0][s;1] \qquad\quad + [s;0][s;0]$

$[1] \rightarrow \qquad\qquad [1][0] \qquad\qquad\quad + [0][1]$

$[1^2] \rightarrow \qquad\qquad [1^2][0] \qquad\qquad\quad + [1][1] \qquad\qquad\quad + [0][1^2]$

$[1^3] \rightarrow \qquad\qquad [1^3][0] \qquad\qquad\quad + [1^2][1] \qquad\qquad + [1][1^2] \qquad\qquad + [0][1^3]$

$[1^4] \rightarrow \qquad\qquad [1^4][0] \qquad\qquad\quad + [1^3][1] \qquad\qquad + [1^2][1^2] \qquad\quad + [1][1^3]$
$\qquad\qquad\qquad\;\; + [0][1^3]$

$[1^5] \rightarrow \qquad\qquad [1^4][1] \qquad\qquad\quad + [1^4][0] \qquad\qquad + [1^3][1^2] \qquad\quad + [1^2][1^3]$
$\qquad\qquad\qquad\;\; + [1][1^3] \qquad\qquad + [0][1^2]$

$[1^6] \rightarrow \qquad\qquad [1^4][1^2] \qquad\qquad + [1^4][1] \qquad\qquad + [1^3][1^3] \qquad\quad + [1^3][0]$
$\qquad\qquad\qquad\;\; + [1^2][1^3] \qquad\quad + [1][1^2] \qquad\qquad + [0][1]$

$[1^7] \rightarrow \qquad\qquad [1^4][1^3] \qquad\qquad + [1^4][1^2] \qquad\quad + [1^3][1^3] \qquad\quad + [1^3][1]$
$\qquad\qquad\qquad\;\; + [1^2][1^2] \qquad\quad + [1^2][0] \qquad\qquad + [1][1] \qquad\qquad\quad + [0][0]$

$[1^8]_+ \rightarrow \qquad\quad [1^4][1^3] \qquad\qquad + [1^3][1^2] \qquad\quad + [1^2][1] \qquad\qquad + [1][0]$

$[2] \rightarrow \qquad\qquad [2][0] \qquad\qquad\quad + [1][1] \qquad\qquad\quad + [0][2] \qquad\qquad\quad + [0][0]$

$[21] \rightarrow \qquad\qquad [21][0] \qquad\qquad\quad + [2][1] \qquad\qquad\quad + [1^2][1] \qquad\qquad + [1][2]$
$\qquad\qquad\qquad\;\; + [1][1^2] \qquad\qquad + [1][0] \qquad\qquad\quad + [0][21] \qquad\qquad + [0][1]$

$[21^2] \rightarrow$

$[21^2][0]$ $+ [21][1]$ $+ [2][1^2]$ $+ [1^3][1]$
$+ [1^2][2]$ $+ [1^2][1^2]$ $+ [1^2][0]$ $+ [1][21]$
$+ [1][1^3]$ $+ [1][1]$ $+ [0][21^2]$ $+ [0][1^2]$

$[21^3] \rightarrow$

$[21^3][0]$ $+ [21^2][1]$ $+ [21][1^2]$ $+ [2][1^3]$
$+ [1^4][1]$ $+ [1^3][2]$ $+ [1^3][1^2]$ $+ [1^3][0]$
$+ [1^2][21]$ $+ [1^2][1^3]$ $+ [1^2][1]$ $+ [1][21^2]$
$+ [1][1^3]$ $+ [1][1^2]$ $+ [0][21^2]$ $+ [0][1^3]$

$[21^4] \rightarrow$

$[21^3][1]$ $+ [21^3][0]$ $+ [21^2][1^2]$ $+ [21][1^3]$
$+ [2][1^3]$ $+ [1^4][2]$ $+ [1^4][1^2]$ $+ [1^4][1]$
$+ [1^4][0]$ $+ [1^3][21]$ $+ [1^3][1^3]$ $+ [1^3][1]$
$+ [1^2][21^2]$ $+ [1^2][1^3]$ $+ [1^2][1^2]$ $+ [1][21^2]$
$+ [1][1^3]$ $+ [1][1^2]$ $+ [0][21]$ $+ [0][1^3]$

$[21^6] \rightarrow$

$[21^3][1^3]$ $+ [21^3][1^2]$ $+ [21^2][1^3]$ $+ [21^2][1]$
$+ [21][1^2]$ $+ [21][0]$ $+ [2][1]$ $+ [1^4][21^2]$
$+ [1^4][21]$ $+ 2[1^4][1^3]$ $+ [1^4][1^2]$ $+ [1^4][1]$
$+ [1^3][21^2]$ $+ [1^3][2]$ $+ [1^3][1^3]$ $+ 2[1^3][1^2]$
$+ [1^3][0]$ $+ [1^2][21]$ $+ [1^2][1^3]$ $+ 2[1^2][1]$
$+ [1][2]$ $+ [1][1^2]$ $+ [1][0]$ $+ [0][1]$

$[2^2] \rightarrow$

$[2^2][0]$ $+ [21][1]$ $+ [2][2]$ $+ [2][0]$
$+ [1^2][1^2]$ $+ [1][21]$ $+ [1][1]$ $+ [0][2^2]$
$+ [0][2]$ $+ [0][0]$

$[2^2 1] \rightarrow$

$[2^2 1][0]$ $+ [2^2][1]$ $+ [21^2][1]$ $+ [21][2]$
$+ [21][1^2]$ $+ [21][0]$ $+ [2][21]$ $+ [2][1]$
$+ [1^3][1^2]$ $+ [1^2][21]$ $+ [1^2][1^3]$ $+ [1^2][1]$
$+ [1][2^2]$ $+ [1][21^2]$ $+ [1][2]$ $+ [1][1^2]$
$+ [1][0]$ $+ [0][2^2 1]$ $+ [0][21]$ $+ [0][1]$

$[2^2 1^2] \rightarrow$

$[2^2 1^2][0]$ $+ [2^2 1][1]$ $+ [2^2][1^2]$ $+ [21^3][1]$
$+ [21^2][2]$ $+ [21^2][1^2]$ $+ [21^2][0]$ $+ [21][21]$
$+ [21][1^3]$ $+ [21][1]$ $+ [2][21^2]$ $+ [2][1^2]$
$+ [1^4][1^2]$ $+ [1^3][21]$ $+ [1^3][1^3]$ $+ [1^3][1]$
$+ [1^2][2^2]$ $+ [1^2][21^2]$ $+ [1^2][2]$ $+ [1^2][1^3]$
$+ [1^2][1^2]$ $+ [1^2][0]$ $+ [1][2^2 1]$ $+ [1][21^2]$
$+ [1][21]$ $+ [1][1^3]$ $+ [1][1]$ $+ [0][2^2 1]$
$+ [0][21^2]$ $+ [0][1^2]$

$[s;2]_+ \rightarrow$

$[s;2][s;0]$ $+ [s;1][s;1]$ $+ [s;1][s;0]$ $+ [s;0][s;2]$
$+ [s;0][s;1]$ $+ [s;0][s;0]$

$[s;21]_+ \rightarrow$

$[s;21][s;0]$ $+ [s;2][s;1]$ $+ [s;2][s;0]$ $+ [s;1^2][s;1]$
$+ [s;1^2][s;0]$ $+ [s;1][s;2]$ $+ [s;1][s;1^2]$ $+ 2[s;1][s;1]$
$+ 2[s;1][s;0]$ $+ [s;0][s;21]$ $+ [s;0][s;2]$ $+ [s;0][s;1^2]$
$+ 2[s;0][s;1]$ $+ [s;0][s;0]$

$$SO_{16} \rightarrow SO_8 \times SO_8 \text{ Branching Rules}$$

$(0) \rightarrow$ $\quad$ $[0][0]$

$(1) \rightarrow$ $\quad$ $[1][0]$ $\quad$ $+ [0][1]$

$(1^2) \rightarrow$ $\quad$ $[1^2][0]$ $\quad$ $+ [1][1]$ $\quad$ $+ [0][1^2]$

$(1^3) \rightarrow$ $\quad$ $[1^3][0]$ $\quad$ $+ [1^2][1]$ $\quad$ $+ [1][1^2]$ $\quad$ $+ [0][1^3]$

$(1^4) \rightarrow$ $\quad$ $[1^4]_+[0]$ $\quad$ $+ [1^4]_-[0]$ $\quad$ $+ [1^3][1]$ $\quad$ $+ [1^2][1^2]$ $\quad$ $+ [1][1^3]$
$\qquad\quad$ $+ [0][1^4]_+$ $\quad$ $+ [0][1^4]_-$

$(1^5) \rightarrow$ $\quad$ $[1^4]_+[1]$ $\quad$ $+ [1^4]_-[1]$ $\quad$ $+ [1^3][1^2]$ $\quad$ $+ [1^3][0]$ $\quad$ $+ [1^2][1^3]$
$\qquad\quad$ $+ [1][1^4]_+$ $\quad$ $+ [1][1^4]_-$ $\quad$ $+ [0][1^3]$

$(1^6) \rightarrow$ $\quad$ $[1^4]_+[1^2]$ $\quad$ $+ [1^4]_-[1^2]$ $\quad$ $+ [1^3][1^3]$ $\quad$ $+ [1^3][1]$ $\quad$ $+ [1^2][1^4]_+$
$\qquad\quad$ $+ [1^2][1^4]_-$ $\quad$ $+ [1^2][0]$ $\quad$ $+ [1][1^3]$ $\quad$ $+ [0][1^2]$

$(1^7) \rightarrow$ $\quad$ $[1^4]_+[1^3]$ $\quad$ $+ [1^4]_-[1^3]$ $\quad$ $+ [1^3][1^4]_+$ $\quad$ $+ [1^3][1^4]_-$ $\quad$ $+ [1^3][1^2]$
$\qquad\quad$ $+ [1^2][1^3]$ $\quad$ $+ [1^2][1]$ $\quad$ $+ [1][1^2]$ $\quad$ $+ [1][0]$ $\quad$ $+ [0][1]$

$(1^8)_+ \rightarrow$ $\quad$ $[1^4]_+[1^4]_+$ $\quad$ $+ [1^4]_-[1^4]_-$ $\quad$ $+ [1^3][1^3]$ $\quad$ $+ [1^2][1^2]$ $\quad$ $+ [1][1]$
$\qquad\quad$ $+ [0][0]$

$(2) \rightarrow$ $\quad$ $[2][0]$ $\quad$ $+ [1][1]$ $\quad$ $+ [0][2]$ $\quad$ $+ [0][0]$

$(21) \rightarrow$ $\quad$ $[21][0]$ $\quad$ $+ [2][1]$ $\quad$ $+ [1^2][1]$ $\quad$ $+ [1][2]$ $\quad$ $+ [1][1^2]$
$\qquad\quad$ $+ [1][0]$ $\quad$ $+ [0][21]$ $\quad$ $+ [0][1]$

$(21^2) \rightarrow$ $\quad$ $[21^2][0]$ $\quad$ $+ [21][1]$ $\quad$ $+ [2][1^2]$ $\quad$ $+ [1^3][1]$ $\quad$ $+ [1^2][2]$
$\qquad\quad$ $+ [1^2][1^2]$ $\quad$ $+ [1^2][0]$ $\quad$ $+ [1][21]$ $\quad$ $+ [1][1^3]$ $\quad$ $+ [1][1]$
$\qquad\quad$ $+ [0][21^2]$ $\quad$ $+ [0][1^2]$

$(21^3) \rightarrow$ $\quad$ $[21^3]_+[0]$ $\quad$ $+ [21^3]_-[0]$ $\quad$ $+ [21^2][1]$ $\quad$ $+ [21][1^2]$ $\quad$ $+ [2][1^3]$
$\qquad\quad$ $+ [1^4]_+[1]$ $\quad$ $+ [1^4]_-[1]$ $\quad$ $+ [1^3][2]$ $\quad$ $+ [1^3][1^2]$ $\quad$ $+ [1^3][0]$
$\qquad\quad$ $+ [1^2][21]$ $\quad$ $+ [1^2][1^3]$ $\quad$ $+ [1^2][1]$ $\quad$ $+ [1][21^2]$ $\quad$ $+ [1][1^4]_+$
$\qquad\quad$ $+ [1][1^4]_-$ $\quad$ $+ [1][1^2]$ $\quad$ $+ [0][21^3]_+$ $\quad$ $+ [0][21^3]_-$ $\quad$ $+ [0][1^3]$

$(21^4) \rightarrow$ $\quad$ $[21^3]_+[1]$ $\quad$ $+ [21^3]_-[1]$ $\quad$ $+ [21^2][1^2]$ $\quad$ $+ [21^2][0]$ $\quad$ $+ [21][1^3]$
$\qquad\quad$ $+ [2][1^4]_+$ $\quad$ $+ [2][1^4]_-$ $\quad$ $+ [1^4]_+[2]$ $\quad$ $+ [1^4]_+[1^2]$ $\quad$ $+ [1^4]_+[0]$
$\qquad\quad$ $+ [1^4]_-[2]$ $\quad$ $+ [1^4]_-[1^2]$ $\quad$ $+ [1^4]_-[0]$ $\quad$ $+ [1^3][21]$ $\quad$ $+ [1^3][1^3]$
$\qquad\quad$ $+ 2[1^3][1]$ $\quad$ $+ [1^2][21^2]$ $\quad$ $+ [1^2][1^4]_+$ $\quad$ $+ [1^2][1^4]_-$ $\quad$ $+ [1^2][1^2]$
$\qquad\quad$ $+ [1][21^3]_+$ $\quad$ $+ [1][21^3]_-$ $\quad$ $+ 2[1][1^3]$ $\quad$ $+ [0][21^2]$ $\quad$ $+ [0][1^4]_+$
$\qquad\quad$ $+ [0][1^4]_-$

$(21^5) \rightarrow$ $\quad$ $[21^3]_+[1^2]$ $\quad$ $+ [21^3]_-[1^2]$ $\quad$ $+ [21^2][1^3]$ $\quad$ $+ [21^2][1]$ $\quad$ $+ [21][1^4]_+$
$\qquad\quad$ $+ [21][1^4]_-$ $\quad$ $+ [21][0]$ $\quad$ $+ [2][1^3]$ $\quad$ $+ [1^4]_+[21]$ $\quad$ $+ [1^4]_+[1^3]$
$\qquad\quad$ $+ [1^4]_+[1]$ $\quad$ $+ [1^4]_-[21]$ $\quad$ $+ [1^4]_-[1^3]$ $\quad$ $+ [1^4]_-[1]$ $\quad$ $+ [1^3][21^2]$
$\qquad\quad$ $+ [1^3][2]$ $\quad$ $+ [1^3][1^4]_+$ $\quad$ $+ [1^3][1^4]_-$ $\quad$ $+ 2[1^3][1^2]$ $\quad$ $+ [1^3][0]$
$\qquad\quad$ $+ [1^2][21^3]_+$ $\quad$ $+ [1^2][21^3]_-$ $\quad$ $+ 2[1^2][1^3]$ $\quad$ $+ [1^2][1]$ $\quad$ $+ [1][21^2]$
$\qquad\quad$ $+ [1][1^4]_+$ $\quad$ $+ [1][1^4]_-$ $\quad$ $+ [1][1^2]$ $\quad$ $+ [0][21]$ $\quad$ $+ [0][1^3]$

$(21^6) \rightarrow$

$[21^3]_+[1^3]$ $+ [21^3]_-[1^3]$ $+ [21^2][1^4]_+$ $+ [21^2][1^4]_-$ $+ [21^2][1^2]$
$+ [21][1^3]$ $+ [21][1]$ $+ [2][1^2]$ $+ [2][0]$ $+ [1^4]_+[21^2]$
$+ [1^4]_+[1^4]_+$ $+ [1^4]_+[1^4]_-$ $+ [1^4]_+[1^2]$ $+ [1^4]_-[21^2]$ $+ [1^4]_-[1^4]_+$
$+ [1^4]_-[1^4]_-$ $+ [1^4]_-[1^2]$ $+ [1^3][21^3]_+$ $+ [1^3][21^3]_-$ $+ [1^3][21]$
$+ 3[1^3][1^3]$ $+ [1^3][1]$ $+ [1^2][21^2]$ $+ [1^2][2]$ $+ [1^2][1^4]_+$
$+ [1^2][1^4]_-$ $+ 2[1^2][1^2]$ $+ [1^2][0]$ $+ [1][21]$ $+ [1][1^3]$
$+ 2[1][1]$ $+ [0][2]$ $+ [0][1^2]$

$(21^7)_+ \rightarrow$

$[21^3]_+[1^4]_+$ $+ [21^3]_-[1^4]_-$ $+ [21^2][1^3]$ $+ [21][1^2]$ $+ [2][1]$
$+ [1^4]_+[21^3]_+$ $+ [1^4]_+[1^3]$ $+ [1^4]_-[21^3]_-$ $+ [1^4]_-[1^3]$ $+ [1^3][21^2]$
$+ [1^3][1^4]_+$ $+ [1^3][1^4]_-$ $+ [1^3][1^2]$ $+ [1^2][21]$ $+ [1^2][1^3]$
$+ [1^2][1]$ $+ [1][2]$ $+ [1][1^2]$ $+ [1][0]$ $+ [0][1]$

$(2^2) \rightarrow$

$[2^2][0]$ $+ [21][1]$ $+ [2][2]$ $+ [2][0]$ $+ [1^2][1^2]$
$+ [1][21]$ $+ [1][1]$ $+ [0][2^2]$ $+ [0][2]$ $+ [0][0]$

$(2^2 1) \rightarrow$

$[2^2 1][0]$ $+ [2^2][1]$ $+ [21^2][1]$ $+ [21][2]$ $+ [21][1^2]$
$+ [21][0]$ $+ [2][21]$ $+ [2][1]$ $+ [1^3][1^2]$ $+ [1^2][21]$
$+ [1^2][1^3]$ $+ [1^2][1]$ $+ [1][2^2]$ $+ [1][21^2]$ $+ [1][2]$
$+ [1][1^2]$ $+ [1][0]$ $+ [0][2^2 1]$ $+ [0][21]$ $+ [0][1]$

$(2^2 1^2) \rightarrow$

$[2^2 1^2]_+[0]$ $+ [2^2 1^2]_-[0]$ $+ [2^2 1][1]$ $+ [2^2][1^2]$ $+ [21^3]_+[1]$
$+ [21^3]_-[1]$ $+ [21^2][2]$ $+ [21^2][1^2]$ $+ [21^2][0]$ $+ [21][21]$
$+ [21][1^3]$ $+ [21][1]$ $+ [2][21^2]$ $+ [2][1^2]$ $+ [1^4]_+[1^2]$
$+ [1^4]_-[1^2]$ $+ [1^3][21]$ $+ [1^3][1^3]$ $+ [1^3][1]$ $+ [1^2][2^2]$
$+ [1^2][21^2]$ $+ [1^2][2]$ $+ [1^2][1^4]_+$ $+ [1^2][1^4]_-$ $+ [1^2][1^2]$
$+ [1^2][0]$ $+ [1][2^2 1]$ $+ [1][21^3]_+$ $+ [1][21^3]_-$ $+ [1][21]$
$+ [1][1^3]$ $+ [1][1]$ $+ [0][2^2 1^2]_+$ $+ [0][2^2 1^2]_-$ $+ [0][21^2]$
$+ [0][1^2]$

$(s;0)_+ \rightarrow \quad [s;0]_+[s;0]_+ \quad\quad + [s;0]_-[s;0]_-$

$(s;1)_+ \rightarrow \quad [s;1]_+[s;0]_+ \quad\quad + [s;1]_-[s;0]_- \quad + [s;0]_+[s;1]_+ \quad + [s;0]_+[s;0]_- \quad + [s;0]_-[s;1]_-$
$\quad\quad\quad\quad\quad + [s;0]_-[s;0]_+$

$(s;1^2)_+ \rightarrow \quad [s;1^2]_+[s;0]_+ \quad + [s;1^2]_-[s;0]_- \quad + [s;1]_+[s;1]_+ \quad + [s;1]_+[s;0]_- \quad + [s;1]_-[s;1]_-$
$\quad\quad\quad\quad\quad + [s;1]_-[s;0]_+ \quad + [s;0]_+[s;1^2]_+ \quad + [s;0]_+[s;1]_- \quad + [s;0]_+[s;0]_+ \quad + [s;0]_-[s;1^2]_-$
$\quad\quad\quad\quad\quad + [s;0]_-[s;1]_+ \quad + [s;0]_-[s;0]_-$

$(s;1^3)_+ \rightarrow \quad [s;1^3]_+[s;0]_+ \quad + [s;1^3]_-[s;0]_- \quad + [s;1^2]_+[s;1]_+ \quad + [s;1^2]_+[s;0]_- \quad + [s;1^2]_-[s;1]_-$
$\quad\quad\quad\quad\quad + [s;1^2]_-[s;0]_+ \quad + [s;1]_+[s;1^2]_+ \quad + [s;1]_+[s;1]_- \quad + [s;1]_+[s;0]_+ \quad + [s;1]_-[s;1^2]_-$
$\quad\quad\quad\quad\quad + [s;1]_-[s;1]_+ \quad + [s;1]_-[s;0]_- \quad + [s;0]_+[s;1^3]_+ \quad + [s;0]_+[s;1^2]_- \quad + [s;0]_+[s;1]_+$
$\quad\quad\quad\quad\quad + [s;0]_+[s;0]_- \quad + [s;0]_-[s;1^3]_- \quad + [s;0]_-[s;1^2]_+ \quad + [s;0]_-[s;1]_- \quad + [s;0]_-[s;0]_+$

$$(s;1^4)_+ \rightarrow$$

$$[s;1^4]_+[s;0]_+ \quad + [s;1^4]_-[s;0]_- \quad + [s;1^3]_+[s;1]_+ \quad + [s;1^3]_+[s;0]_- \quad + [s;1^3]_-[s;1]_-$$
$$+ [s;1^3]_-[s;0]_+ \quad + [s;1^2]_+[s;1^2]_+ \quad + [s;1^2]_+[s;1]_- \quad + [s;1^2]_+[s;0]_+ \quad + [s;1^2]_-[s;1^2]_-$$
$$+ [s;1^2]_-[s;1]_+ \quad + [s;1^2]_-[s;0]_- \quad + [s;1]_+[s;1^3]_+ \quad + [s;1]_+[s;1^2]_- \quad + [s;1]_+[s;1]_+$$
$$+ [s;1]_+[s;0]_- \quad + [s;1]_-[s;1^3]_- \quad + [s;1]_-[s;1^2]_+ \quad + [s;1]_-[s;1]_- \quad + [s;1]_-[s;0]_+$$
$$+ [s;0]_+[s;1^4]_+ \quad + [s;0]_+[s;1^3]_- \quad + [s;0]_+[s;1^2]_+ \quad + [s;0]_+[s;1]_- \quad + [s;0]_+[s;0]_+$$
$$+ [s;0]_-[s;1^4]_- \quad + [s;0]_-[s;1^3]_+ \quad + [s;0]_-[s;1^2]_- \quad + [s;0]_-[s;1]_+ \quad + [s;0]_-[s;0]_-$$

$$(s;2)_+ \rightarrow$$

$$[s;2]_+[s;0]_+ \quad + [s;2]_-[s;0]_- \quad + [s;1]_+[s;1]_+ \quad + [s;1]_+[s;0]_- \quad + [s;1]_-[s;1]_-$$
$$+ [s;1]_-[s;0]_+ \quad + [s;0]_+[s;2]_+ \quad + [s;0]_+[s;1]_- \quad + [s;0]_+[s;0]_+ \quad + [s;0]_-[s;2]_-$$
$$+ [s;0]_-[s;1]_+ \quad + [s;0]_-[s;0]_-$$

$$(s;21)_+ \rightarrow$$

$$[s;21]_+[s;0]_+ \quad + [s;21]_-[s;0]_- \quad + [s;2]_+[s;1]_+ \quad + [s;2]_+[s;0]_- \quad + [s;2]_-[s;1]_-$$
$$+ [s;2]_-[s;0]_+ \quad + [s;1^2]_+[s;1]_+ \quad + [s;1^2]_+[s;0]_- \quad + [s;1^2]_-[s;1]_- \quad + [s;1^2]_-[s;0]_+$$
$$+ [s;1]_+[s;2]_+ \quad + [s;1]_+[s;1^2]_+ \quad + 2[s;1]_+[s;1]_- \quad + 2[s;1]_+[s;0]_+ \quad + [s;1]_-[s;2]_-$$
$$+ [s;1]_-[s;1^2]_- \quad + 2[s;1]_-[s;1]_+ \quad + 2[s;1]_-[s;0]_- \quad + [s;0]_+[s;21]_+ \quad + [s;0]_+[s;2]_-$$
$$+ [s;0]_+[s;1^2]_- \quad + 2[s;0]_+[s;1]_+ \quad + [s;0]_+[s;0]_- \quad + [s;0]_-[s;21]_- \quad + [s;0]_-[s;2]_+$$
$$+ [s;0]_-[s;1^2]_+ \quad + 2[s;0]_-[s;1]_- \quad + [s;0]_-[s;0]_+$$

$$(s;21^2)_+ \rightarrow$$

$$[s;21^2]_+[s;0]_+ \quad + [s;21^2]_-[s;0]_- \quad + [s;21]_+[s;1]_+ \quad + [s;21]_+[s;0]_- \quad + [s;21]_-[s;1]_-$$
$$+ [s;21]_-[s;0]_+ \quad + [s;2]_+[s;1^2]_+ \quad + [s;2]_+[s;1]_- \quad + [s;2]_+[s;0]_+ \quad + [s;2]_-[s;1^2]_-$$
$$+ [s;2]_-[s;1]_+ \quad + [s;2]_-[s;0]_- \quad + [s;1^3]_+[s;1]_+ \quad + [s;1^3]_+[s;0]_- \quad + [s;1^3]_-[s;1]_-$$
$$+ [s;1^3]_-[s;0]_+ \quad + [s;1^2]_+[s;2]_+ \quad + [s;1^2]_+[s;1^2]_+ \quad + 2[s;1^2]_+[s;1]_- \quad + 2[s;1^2]_+[s;0]_+$$
$$+ [s;1^2]_-[s;2]_- \quad + [s;1^2]_-[s;1^2]_- \quad + 2[s;1^2]_-[s;1]_+ \quad + 2[s;1^2]_-[s;0]_- \quad + [s;1]_+[s;21]_+$$
$$+ [s;1]_+[s;2]_- \quad + [s;1]_+[s;1^3]_+ \quad + 2[s;1]_+[s;1^2]_- \quad + 3[s;1]_+[s;1]_+ \quad + 2[s;1]_+[s;0]_-$$
$$+ [s;1]_-[s;21]_- \quad + [s;1]_-[s;2]_+ \quad + [s;1]_-[s;1^3]_- \quad + 2[s;1]_-[s;1^2]_+ \quad + 3[s;1]_-[s;1]_-$$
$$+ 2[s;1]_-[s;0]_+ \quad + [s;0]_+[s;21^2]_+ \quad + [s;0]_+[s;21]_- \quad + [s;0]_+[s;2]_+ \quad + [s;0]_+[s;1^3]_-$$
$$+ 2[s;0]_+[s;1^2]_+ \quad + 2[s;0]_+[s;1]_- \quad + [s;0]_+[s;0]_+ \quad + [s;0]_-[s;21^2]_- \quad + [s;0]_-[s;21]_+$$
$$+ [s;0]_-[s;2]_- \quad + [s;0]_-[s;1^3]_+ \quad + 2[s;0]_-[s;1^2]_- \quad + 2[s;0]_-[s;1]_+ \quad + [s;0]_-[s;0]_-$$

$$(s;2^2)_+ \rightarrow$$

$$[s;2^2]_+[s;0]_+ \quad + [s;2^2]_-[s;0]_- \quad + [s;21]_+[s;1]_+ \quad + [s;21]_+[s;0]_- \quad + [s;21]_-[s;1]_-$$
$$+ [s;21]_-[s;0]_+ \quad + [s;2]_+[s;2]_+ \quad + [s;2]_+[s;1]_- \quad + [s;2]_+[s;0]_+ \quad + [s;2]_-[s;2]_-$$
$$+ [s;2]_-[s;1]_+ \quad + [s;2]_-[s;0]_- \quad + [s;1^2]_+[s;1^2]_+ \quad + [s;1^2]_+[s;1]_- \quad + [s;1^2]_+[s;0]_+$$
$$+ [s;1^2]_-[s;1^2]_- \quad + [s;1^2]_-[s;1]_+ \quad + [s;1^2]_-[s;0]_- \quad + [s;1]_+[s;21]_+ \quad + [s;1]_+[s;2]_-$$
$$+ [s;1]_+[s;1^2]_- \quad + 2[s;1]_+[s;1]_+ \quad + [s;1]_+[s;0]_- \quad + [s;1]_-[s;21]_- \quad + [s;1]_-[s;2]_+$$
$$+ [s;1]_-[s;1^2]_+ \quad + 2[s;1]_-[s;1]_- \quad + [s;1]_-[s;0]_+ \quad + [s;0]_+[s;2^2]_+ \quad + [s;0]_+[s;21]_-$$
$$+ [s;0]_+[s;2]_+ \quad + [s;0]_+[s;1^2]_+ \quad + [s;0]_+[s;1]_- \quad + [s;0]_+[s;0]_+ \quad + [s;0]_-[s;2^2]_-$$
$$+ [s;0]_-[s;21]_+ \quad + [s;0]_-[s;2]_- \quad + [s;0]_-[s;1^2]_- \quad + [s;0]_-[s;1]_+ \quad + [s;0]_-[s;0]_-$$