SCHUR based Conjectures

One of the most useful features of SCHUR is the establishment of conjectures. From time to time I will place here new conjectures with the hope that someone will prove or disprove them.

The KW Plethysm Conjecture

In the notation of Littlewood Suppose $\{\lambda\}$ denotes a non-trivial $S-$function associated with the partition (λ) the we conjecture that:-

If $\{\lambda\}^p$ *contains the* $S-$*function* $\{\mu\}$ $n-$*times and* $p >= 3$ *then if*

$$\{\lambda\} \otimes \{p\} \supset s\{\mu\}$$
$$\{\lambda\} \otimes \{1^p\} \supset a\{\mu\}$$

then $a, s < n$.

Conjecture 2

Let D be the infinite series of $S-$functions having all parts even. Let D_k be those members of the D series with not more than k parts. We conjecture that:-

$$D_k \otimes \{2\} = \sum_{i=0}^{\infty} \{D_k^{4i} \cdot D_k\} \tag{1}$$

$$D_k \otimes \{1^2\} = \sum_{i=0}^{\infty} \{D_k^{4i+2} \cdot D_k\} \tag{2}$$

Solution to Conjecture 2

This follows by noting that

$$D = \{2\} \otimes M$$
$$\sum_{i=0}^{\infty} D^{4i} = \{2\} \otimes M_+$$
$$D \otimes \{2\} = \{2\} \otimes M \otimes \{2\}$$

But $M \otimes \{2\} = M\dot{M}_+$ and hence

$$D \otimes \{2\} = \{2\} \otimes M\dot{M}_+$$

But

$$\sum_{i=0}^{\infty} D^{4i} \dot{D} = \{2\} \otimes M_+ \dot{\{2\}} \otimes M$$
$$= \{2\} \otimes M\dot{M}_+$$

and hence we are led to

$$D_k \otimes \{2\} = \sum_{i=0}^{\infty} \{D_k^{4i} \cdot D_k\}$$

The second part of the conjecture follows in the same manner after noting that

$$M \otimes \{1^2\} = M\dot{M}_-$$

Conjecture 3 Kronecker Products in $Sp(2n, R)$

1. The harmonic discrete series irreps of $Sp(2n, R)$ may be uniquely labelled as $< \frac{k}{2}; (\lambda) >$ where the partitions (λ) are restricted by the requirement that

$$\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq k \tag{1}$$
$$\tilde{\lambda}_1 \leq n \tag{2}$$

2. Eqs.(1) and (2) restrict the length ℓ_λ of the partition (λ). If $n \geq \ell_\lambda > k$ then the partition is non-standard in $O(k)$. The modification rules for $O(k)$ give

$$[\lambda] = (-1)^{x-1}[\lambda - h]^* \text{where } h = 2\ell_\lambda - k \tag{3}$$

where x is the last column used in the removal of the continuous rim hook of length h. Given the restrictions of Eqs. (1) and (2) we are limited to $x = 1$ and removing $k - \tilde{\lambda}_1$ boxes from the first column to leave an associated irrep of $O(k)$. Conversely, starting with a standard irrep $[\lambda]$ of $O(k)$ we may form a non-standard irrep of $O(k)$ by adding $h_\lambda = k - 2\ell_\lambda$ boxes to the foot of the first column of (λ). This will give a standard labelled irrep of $Sp(2n, R)$ though the partition $(\lambda + h_\lambda)$ is $O(k)$ non-standard. For brievity we shall write

$$[\lambda + h_\lambda] \equiv [\lambda_h] \tag{4}$$

Under $O(k)$ standardisation

$$[\lambda_h] \rightarrow [\lambda]^* \tag{5}$$

.

3. Let us now define

$$< \frac{k}{2}; (\lambda) >^* = < \frac{k}{2}; (\lambda)^* > \tag{6}$$

4. From KW (8.15) (This refers to publication 95) we have

$$< \frac{k}{2}; (\mu) > \times < \frac{\ell}{2}; (\nu) > = \sum_\lambda R^{\mu\nu}_\lambda < \frac{k+\ell}{2}; (\lambda) > \tag{7}$$

where the coefficients $R^{\mu\nu}_\lambda$ are the branching rule coefficients appropriate to the restriction $O(k+\ell) \rightarrow O(k) \times O(\ell)$. Since $(R^{\mu\nu}_\lambda)^* = R^{\mu\nu}_\lambda$ we have

$$(< \frac{k}{2}; (\mu) > \times < \frac{\ell}{2}; (\nu) >)^* = < \frac{k}{2}; (\mu)^* > \times < \frac{\ell}{2}; (\nu)^* > \tag{8}$$

5. Let us define operators $\mathcal{M}$ and $\mathcal{M}^{-1}$ such that

$$\mathcal{M} < \frac{k}{2}; (\lambda_h) > = < \frac{k}{2}; (\lambda) > \tag{8a}$$

$$\mathcal{M}^{-1} < \frac{k}{2}; (\lambda) > = < \frac{k}{2}; (\lambda_h) > \tag{8b}$$

Conjectures on Kronecker products in $Sp(2n, R)$

If $< \frac{k}{2}; (\mu) > = (\mathcal{M}$ or $\mathcal{M}^{-1})(< \frac{k}{2}; (\mu') >)$ and likewise for $< \frac{\ell}{2}; (\nu) >$ then for sufficiently large n

$$\mathcal{M}(< \frac{k}{2}; (\mu) >< \frac{\ell}{2}; (\nu) >) = \mathcal{M}(< \frac{k}{2}; (\mu') >< \frac{\ell}{2}; (\nu') >) \tag{9a}$$

$$\mathcal{M}^{-1}(< \frac{k}{2}; (\mu) >< \frac{\ell}{2}; (\nu) >) = \mathcal{M}^{-1}(< \frac{k}{2}; (\mu') >< \frac{\ell}{2}; (\nu') >) \tag{9b}$$

By sufficiently large n we mean large enough so that no $Sp(2n, R)$ modification rules are needed. In every case it is to be understood that $\mathcal{M}$ or $\mathcal{M}^{-1}$ acts on the resultant of the Kronecker product.

In certain cases the action of $\mathcal{M}$ or $\mathcal{M}^{-1}$ on one side or the other will have no effect and the conjecture simplifies.

Conjectures on Kronecker Powers in $Sp(2n, R)$

Considerations of the squares of irreps of $Sp(2n, R)$ strongly suggest the following conjecture:-

$$M(< \frac{k}{2}; (1^k) >^2) = < \frac{k}{2}; (0) >^2 \tag{10}$$

Limited data on cubes would suggest that Eq.(10) generalises to

$$M(< \frac{k}{2}; (1^k) >^p) = M(< \frac{k}{2}; (0) >^p) \tag{11}$$

and further that

$$M(< \frac{k}{2};(\lambda_h) >^p) = M(< \frac{k}{2};(\lambda) >^p) \tag{12}$$

Some examples for $Sp(18,R)$

In these examples we use $\mathcal{M}$ though they are equally valid for $\mathcal{M}^{-1}$. If $\mathcal{M}$ is on the left-hand-side only then the other case has $\mathcal{M}^{-1}$ on the right-hand-side.

$$\mathcal{M}(< s;(0) >< 1;(1) >) = \mathcal{M}(< s;(1) >< 1;(1) >) \tag{13a}$$

$$\mathcal{M}(< s;(0) >< 1;(1^2) >) =< s;(1) >< 1;(0) > \tag{13b}$$

$$\mathcal{M}(< 1;(1^2) >< 1;(1^2) >) =< 1;(0) >< 1;(0) > \tag{13c}$$

$$\mathcal{M}(< 1;(1) >< 1;(1^2) >) =< 1;(1) >< 1;(0) > \tag{13d}$$

$$\mathcal{M}(< s1;(31) >< s1;(21) >) = \mathcal{M}(< s1;(3) >< s1;(21) >) \tag{13c}$$

$$\mathcal{M}(< 2;(21^2) >< 1;(0) >) =< 2;(2) >< 1;(1^2) > \tag{13d}$$

$$\mathcal{M}(< 2;(31^2) >< 2;(2) >) = \mathcal{M}(< 2;(3) >< 2;(21^2) >) \tag{13e}$$

$$\mathcal{M}(< 2;(31^2) >< 2;(21^2) >) = \mathcal{M}(< 2;(3) >< 2;(2) >) \tag{13f}$$

$$\mathcal{M}(< 2;(21^2) >< 2;(21^2) >) = \mathcal{M}(< 2;(2) >< 2;(2) > \tag{13g}$$

$$\mathcal{M}(< 2;(1^4) >< 2;(1^4) >) =< 2;(0) >< 2;(0) > \tag{13h}$$

$$\mathcal{M}(< 3;(1^4) >< 3;(1^4) >) =< 3;(1^2) >< 3;(1^2) > \tag{13i}$$

Conjectures on Reduced Inner Plethysms

The relevant references are contained in papers 84 and 132 of Publications.

Suppose X stands for one of the infinite S-fn series designated in 84 as A, C, E, G, H, L, P, R, W. The coefficients in these series carry a phase of ± 1. Then we make the following conjectures concerning the coefficients c^ρ in the expansion of

$$< 1 > \otimes \{\lambda/X\} = \sum_\rho c^\rho < \rho >$$

$$c^\rho >= 0 \quad all\ \lambda \quad for\ X = C, G, L, V \tag{1a}$$

$$c^\rho >= 0 \quad all\ \lambda \supset T \quad for\ X = A, E, P, W \tag{1b}$$

For $X = H, R$ the c^ρ may be ± 1.

The case for $X = G$ is obvious since $< 1 > \otimes\{\lambda/G\}$ enumerates the group subgroup decomposition $O(n)- > S(n)$. Likewise $< 1 > \otimes\{\lambda/C\}$ enumerates the group subgroup decomposition $O(n-1)- > S(n)$ but what do the other examples enumerate?

A Plethysm Conjecture for the $M-$Series

Consider the infinite $S-$function series

$$M = \sum_{m=0}^{\infty} \{m\} \tag{1}$$

with

$$M_+ = \sum_{m=0}^{\infty} \{2m\} \tag{2a}$$

$$M_- = \sum_{m=0}^{\infty} \{2m+1\} \tag{2b}$$

We conjecture that if (λ) is a partition of weight w_λ and $(\tilde{\lambda})$ is the partition conjugate to (λ) then we conjecture that

$$M_+ \otimes \{\lambda\} \supset (M_- \otimes \{\tilde{\lambda}\})/\{1^{w_\lambda}\} \tag{3a}$$

$$M_- \otimes \{\tilde{\lambda}\} \supset (M_+ \otimes \{\lambda\})/\{1^{w_\lambda}\} \tag{3b}$$

The above conjecture would imply that it is sufficent to compute only the w_λ parts of the two plethysms to be then able to deduce all terms involving partitions of fewer than w_λ parts.

Supporting evidence for this conjecture may be seen in the following tables:-

$M_+ \otimes \{3\} =$

$\{0\}$	$+ \{2\}$	$+ \{2^2\}$	$+ \{2^3\}$	$+ 2\{4\}$	$+ 2\{42\}$	$+ \{42^2\}$
$+ 2\{4^2\}$	$+ \{4^22\}$	$+ \{4^3\}$	$+ \{51\}$	$+ \{521\}$	$+ \{53\}$	$+ \{541\}$
$+ 3\{6\}$	$+ 4\{62\}$	$+ 2\{62^2\}$	$+ \{631\}$	$+ 4\{64\}$	$+ 2\{642\}$	$+ \{651\}$
$+ 3\{6^2\}$	$+ 2\{71\}$	$+ 2\{721\}$	$+ 3\{73\}$	$+ \{732\}$	$+ 3\{741\}$	$+ 2\{75\}$
$+ 4\{8\}$	$+ 6\{82\}$	$+ 3\{82^2\}$	$+ 2\{831\}$	$+ 7\{84\}$	$+ 4\{91\}$	$+ 3\{921\}$
$+ 6\{93\}$	$+ 5\{10\ \}$	$+ \{10\ 1^2\}$	$+ 9\{10\ 2\}$	$+ 5\{11\ 1\}$	$+ 7\{12\ \}$	

$M_- \otimes \{1^3\} =$

$\{1^3\}$	$+ \{31^2\}$	$+ \{3^21\}$	$+ \{3^3\}$	$+ \{41\}$	$+ \{43\}$	$+ 2\{51^2\}$
$+ \{52\}$	$+ 2\{531\}$	$+ \{53^2\}$	$+ \{54\}$	$+ 2\{5^21\}$	$+ \{5^23\}$	$+ 2\{61\}$
$+ \{621\}$	$+ 3\{63\}$	$+ \{632\}$	$+ \{641\}$	$+ 2\{65\}$	$+ \{652\}$	$+ 3\{71^2\}$
$+ 2\{72\}$	$+ 4\{731\}$	$+ 2\{73^2\}$	$+ 3\{74\}$	$+ \{742\}$	$+ 4\{751\}$	$+ 2\{76\}$
$+ 3\{81\}$	$+ 2\{821\}$	$+ 5\{83\}$	$+ 2\{832\}$	$+ 3\{841\}$	$+ 5\{85\}$	$+ \{9\}$
$+ 4\{91^2\}$	$+ 4\{92\}$	$+ 6\{931\}$	$+ 6\{94\}$	$+ 5\{10\ 1\}$	$+ 4\{10\ 21\}$	$+ 8\{10\ 3\}$
$+ \{11\ \}$	$+ 5\{11\ 1^2\}$	$+ 6\{11\ 2\}$	$+ 7\{12\ 1\}$	$+ 2\{13\ \}$		

$$M_+ \otimes \{21\} =$$

$\{2\} + \{2^2\} + \{31\} + \{321\} + 2\{4\} + 3\{42\} + \{42^2\}$
$+ \{431\} + 2\{4^2\} + \{4^22\} + 3\{51\} + 2\{521\} + 3\{53\} + \{532\}$
$+ 2\{541\} + \{543\} + \{5^2\} + 3\{6\} + \{61^2\} + 6\{62\} + 2\{62^2\}$
$+ 3\{631\} + 6\{64\} + 3\{642\} + 3\{651\} + 3\{6^2\} + 5\{71\} + 4\{721\}$
$+ 7\{73\} + 3\{732\} + 5\{741\} + 6\{75\} + 5\{8\} + 2\{81^2\} + 10\{82\}$
$+ 3\{82^2\} + 6\{831\} + 12\{84\} + 8\{91\} + 7\{921\} + 12\{93\} + 7\{10\ \}$
$+ 3\{10\ 1^2\} + 15\{10\ 2\} + 12\{11\ 1\} + 9\{12\ \}$

$$M_- \otimes \{21\} =$$

$\{21\} + \{31^2\} + \{32\} + \{3^21\} + 2\{41\} + \{421\} + 2\{43\}$
$+ \{432\} + \{5\} + 2\{51^2\} + 3\{52\} + 3\{531\} + \{53^2\} + 3\{54\}$
$+ \{542\} + 2\{5^21\} + \{5^23\} + 4\{61\} + 3\{621\} + 5\{63\} + 2\{632\}$
$+ 3\{641\} + \{643\} + 4\{65\} + 2\{652\} + \{6^21\} + 2\{7\} + 3\{71^2\}$
$+ 6\{72\} + \{72^2\} + 6\{731\} + 2\{73^2\} + 7\{74\} + 3\{742\} + 6\{751\}$
$+ 5\{76\} + 7\{81\} + 5\{821\} + 10\{83\} + 4\{832\} + 7\{841\} + 10\{85\}$
$+ 3\{9\} + 5\{91^2\} + 10\{92\} + 2\{92^2\} + 10\{931\} + 13\{94\} + 10\{10\ 1\}$
$+ 8\{10\ 21\} + 16\{10\ 3\} + 5\{11\ \} + 7\{11\ 1^2\} + 15\{11\ 2\} + 14\{12\ 1\} + 7\{13\ \}$

$$M_+ \otimes \{4\} =$$

$\{0\}$	$+\ \{2\}$	$+\ \{2^2\}$	$+\ \{2^3\}$	$+\ \{2^4\}$	$+\ 2\{4\}$	$+\ 2\{42\}$
$+\ 2\{42^2\}$	$+\ \{42^3\}$	$+\ 3\{4^2\}$	$+\ 2\{4^22\}$	$+\ \{4^22^2\}$	$+\ 2\{4^3\}$	$+\ \{51\}$
$+\ \{521\}$	$+\ \{52^21\}$	$+\ \{53\}$	$+\ \{532\}$	$+\ 2\{541\}$	$+\ \{5421\}$	$+\ \{543\}$
$+\ \{5^21^2\}$	$+\ 3\{6\}$	$+\ 5\{62\}$	$+\ 4\{62^2\}$	$+\ 2\{62^3\}$	$+\ 2\{631\}$	$+\ \{6321\}$
$+\ 6\{64\}$	$+\ 7\{642\}$	$+\ 3\{651\}$	$+\ 6\{6^2\}$	$+\ 2\{71\}$	$+\ 3\{721\}$	$+\ 2\{72^21\}$
$+\ 5\{73\}$	$+\ \{731^2\}$	$+\ 4\{732\}$	$+\ 7\{741\}$	$+\ 4\{75\}$	$+\ 5\{8\}$	$+\ 8\{82\}$
$+\ 8\{82^2\}$	$+\ 5\{831\}$	$+\ 13\{84\}$	$+\ 5\{91\}$	$+\ 6\{921\}$	$+\ 10\{93\}$	$+\ 6\{10\ \}$
$+\ \{10\ 1^2\}$	$+\ 14\{10\ 2\}$	$+\ 7\{11\ 1\}$	$+\ 9\{12\ \}$			

$$M_- \otimes \{1^4\} =$$

$\{1^4\}$	$+\ \{31^3\}$	$+\ \{3^21^2\}$	$+\ \{3^31\}$	$+\ \{3^4\}$	$+\ \{41^2\}$
$+\ \{431\}$	$+\ \{43^2\}$	$+\ \{4^2\}$	$+\ 2\{51^3\}$	$+\ \{521\}$	$+\ 2\{531^2\}$
$+\ \{532\}$	$+\ 2\{53^21\}$	$+\ \{53^3\}$	$+\ 2\{541\}$	$+\ \{543\}$	$+\ 3\{5^21^2\}$
$+\ \{5^22\}$	$+\ 2\{5^231\}$	$+\ \{5^24\}$	$+\ 2\{61^2\}$	$+\ \{62\}$	$+\ \{621^2\}$
$+\ 4\{631\}$	$+\ \{6321\}$	$+\ 3\{63^2\}$	$+\ \{63^22\}$	$+\ 2\{64\}$	$+\ \{641^2\}$
$+\ 3\{642\}$	$+\ \{6431\}$	$+\ 4\{651\}$	$+\ 2\{6521\}$	$+\ 4\{653\}$	$+\ 3\{6^2\}$
$+\ 2\{6^22\}$	$+\ 3\{71^3\}$	$+\ 3\{721\}$	$+\ 2\{73\}$	$+\ 5\{731^2\}$	$+\ 3\{732\}$
$+\ 4\{73^21\}$	$+\ 7\{741\}$	$+\ 2\{7421\}$	$+\ 5\{743\}$	$+\ 2\{75\}$	$+\ 6\{751^2\}$
$+\ 7\{752\}$	$+\ 7\{761\}$	$+\ \{7^2\}$	$+\ 4\{81^2\}$	$+\ \{82\}$	$+\ 2\{821^2\}$
$+\ \{82^2\}$	$+\ 9\{831\}$	$+\ 3\{8321\}$	$+\ 7\{83^2\}$	$+\ 5\{84\}$	$+\ 5\{841^2\}$
$+\ 7\{842\}$	$+\ 13\{851\}$	$+\ 6\{86\}$	$+\ \{91\}$	$+\ 5\{91^3\}$	$+\ 6\{921\}$
$+\ 4\{93\}$	$+\ 8\{931^2\}$	$+\ 8\{932\}$	$+\ 15\{941\}$	$+\ 8\{95\}$	$+\ 6\{10\ 1^2\}$
$+\ 4\{10\ 2\}$	$+\ 5\{10\ 21^2\}$	$+\ 2\{10\ 2^2\}$	$+\ 17\{10\ 31\}$	$+\ 10\{10\ 4\}$	$+\ 2\{11\ 1\}$
$+\ 6\{11\ 1^3\}$	$+\ 11\{11\ 21\}$	$+\ 9\{11\ 3\}$	$+\ 10\{12\ 1^2\}$	$+\ 6\{12\ 2\}$	$+\ 4\{13\ 1\}$

$$M_+ \otimes \{31\} =$$

$$
\begin{array}{llllll}
\{2\} & + \{2^2\} & + \{2^3\} & + \{31\} & + \{321\} & + \{32^21\} \\
+ 2\{4\} & + 4\{42\} & + 3\{42^2\} & + \{42^3\} & + 2\{431\} & + \{4321\} \\
+ 3\{4^2\} & + 4\{4^22\} & + \{4^22^2\} & + \{4^231\} & + 2\{4^3\} & + \{4^32\} \\
+ 3\{51\} & + 4\{521\} & + 2\{52^21\} & + 5\{53\} & + \{531^2\} & + 4\{532\} \\
+ \{532^2\} & + 6\{541\} & + 3\{5421\} & + 4\{543\} & + \{5432\} & + 2\{54^21\} \\
+ 2\{5^2\} & + \{5^21^2\} & + 3\{5^22\} & + \{5^231\} & + \{5^24\} & + 4\{6\} \\
+ \{61^2\} & + 9\{62\} & + \{621^2\} & + 8\{62^2\} & + 2\{62^3\} & + 8\{631\} \\
+ 4\{6321\} & + 2\{63^2\} & + 13\{64\} & + 3\{641^2\} & + 15\{642\} & + 4\{642^2\} \\
+ 4\{6431\} & + 8\{64^2\} & + 11\{651\} & + 6\{6521\} & + 8\{653\} & + 8\{6^2\} \\
+ 2\{6^21^2\} & + 13\{6^22\} & + 7\{71\} & + 10\{721\} & + 5\{72^21\} & + 13\{73\} \\
+ 3\{731^2\} & + 14\{732\} & + 3\{732^2\} & + 2\{73^21\} & + 20\{741\} & + 11\{7421\} \\
+ 15\{743\} & + 16\{75\} & + 7\{751^2\} & + 22\{752\} & + 20\{761\} & + 6\{7^2\} \\
+ 6\{8\} & + 3\{81^2\} & + 18\{82\} & + 3\{821^2\} & + 15\{82^2\} & + 4\{82^3\} \\
+ 20\{831\} & + 10\{8321\} & + 7\{83^2\} & + 28\{84\} & + 8\{841^2\} & + 38\{842\} \\
+ 34\{851\} & + 31\{86\} & + 12\{91\} & + 20\{921\} & + 9\{92^21\} & + 28\{93\} \\
+ 8\{931^2\} & + 30\{932\} & + 45\{941\} & + 39\{95\} & + 10\{10\ \} & + 7\{10\ 1^2\} \\
+ 30\{10\ 2\} & + 6\{10\ 21^2\} & + 27\{10\ 2^2\} & + 40\{10\ 31\} & + 54\{10\ 4\} & + 20\{11\ 1\} \\
+ \{11\ 1^3\} & + 35\{11\ 21\} & + 48\{11\ 3\} & + 14\{12\ \} & + 12\{12\ 1^2\} & + 48\{12\ 2\} \\
+ 30\{13\ 1\} & + 20\{14\ \} & & & &
\end{array}
$$

$$M_- \otimes \{21^2\} =$$

$$
\begin{array}{llllll}
\{21^2\} & + \{31^3\} & + \{321\} & + \{3^21^2\} & + \{3^22\} & + \{3^31\} \\
+ 2\{41^2\} & + \{42\} & + \{421^2\} & + 3\{431\} & + \{4321\} & + 2\{43^2\} \\
+ \{43^22\} & + \{4^2\} & + \{4^22\} & + \{51\} & + 2\{51^3\} & + 4\{521\} \\
+ 3\{53\} & + 4\{531^2\} & + 4\{532\} & + 3\{53^21\} & + \{53^3\} & + 6\{541\} \\
+ 2\{5421\} & + 4\{543\} & + \{5432\} & + 2\{5^2\} & + 3\{5^21^2\} & + 5\{5^22\} \\
+ 4\{5^231\} & + 3\{5^24\} & + 5\{61^2\} & + 3\{62\} & + 3\{621^2\} & + 2\{62^2\} \\
+ 11\{631\} & + 4\{6321\} & + 7\{63^2\} & + 2\{63^22\} & + 7\{64\} & + 5\{641^2\} \\
+ 8\{642\} & + \{642^2\} & + 4\{6431\} & + 2\{64^2\} & + 13\{651\} & + 6\{6521\} \\
+ 11\{653\} & + 4\{6^2\} & + 2\{6^21^2\} & + 7\{6^22\} & + 3\{71\} & + 4\{71^3\} \\
+ 10\{721\} & + \{72^21\} & + 8\{73\} & + 9\{731^2\} & + 13\{732\} & + \{732^2\} \\
+ 8\{73^21\} & + 20\{741\} & + 8\{7421\} & + 15\{743\} & + 12\{75\} & + 13\{751^2\} \\
+ 22\{752\} & + 20\{761\} & + 6\{7^2\} & + 9\{81^2\} & + 8\{82\} & + 7\{821^2\} \\
+ 5\{82^2\} & + 25\{831\} & + 10\{8321\} & + 16\{83^2\} & + 17\{84\} & + 13\{841^2\} \\
+ 26\{842\} & + 38\{851\} & + 21\{86\} & + 6\{91\} & + 6\{91^3\} & + 20\{921\} \\
+ 3\{92^21\} & + 19\{93\} & + 18\{931^2\} & + 27\{932\} & + 45\{941\} & + 30\{95\} \\
+ \{10\ \} & + 16\{10\ 1^2\} & + 15\{10\ 2\} & + 12\{10\ 21^2\} & + 12\{10\ 2^2\} & + 48\{10\ 31\} \\
+ 36\{10\ 4\} & + 11\{11\ 1\} & + 10\{11\ 1^3\} & + 35\{11\ 21\} & + 34\{11\ 3\} & + 2\{12\ \} \\
+ 24\{12\ 1^2\} & + 27\{12\ 2\} & + 18\{13\ 1\} & + 4\{14\ \} & &
\end{array}
$$

$$M_+ \otimes \{2^2\} =$$

$\{2^2\} + \{321\} + \{3^21^2\} + \{4\} + 2\{42\} + 2\{42^2\}$
$+ \{431\} + \{4321\} + 3\{4^2\} + 2\{4^22\} + \{4^22^2\} + \{4^3\}$
$+ 2\{51\} + 3\{521\} + \{52^21\} + 2\{53\} + \{531^2\} + 3\{532\}$
$+ \{53^21\} + 4\{541\} + 2\{5421\} + 3\{543\} + \{5432\} + \{54^21\}$
$+ 2\{5^2\} + 2\{5^21^2\} + \{5^22\} + \{5^231\} + \{5^24\} + \{6\}$
$+ \{61^2\} + 6\{62\} + \{621^2\} + 4\{62^2\} + \{62^3\} + 6\{631\}$
$+ 3\{6321\} + \{63^2\} + 7\{64\} + \{641^2\} + 10\{642\} + 2\{642^2\}$
$+ 3\{6431\} + 4\{64^2\} + 8\{651\} + 4\{6521\} + 6\{653\} + 6\{6^2\}$
$+ 2\{6^21^2\} + 7\{6^22\} + 3\{71\} + 7\{721\} + 2\{72^21\} + 8\{73\}$
$+ 4\{731^2\} + 9\{732\} + 2\{732^2\} + 2\{73^21\} + 13\{741\} + 7\{7421\}$
$+ 10\{743\} + 8\{75\} + 5\{751^2\} + 15\{752\} + 13\{761\} + 5\{7^2\}$
$+ 3\{8\} + 3\{81^2\} + 10\{82\} + 2\{821^2\} + 9\{82^2\} + \{82^3\}$
$+ 14\{831\} + 7\{8321\} + 6\{83^2\} + 19\{84\} + 6\{841^2\} + 23\{842\}$
$+ 24\{851\} + 18\{86\} + 7\{91\} + \{91^3\} + 14\{921\} + 5\{92^21\}$
$+ 15\{93\} + 7\{931^2\} + 20\{932\} + 30\{941\} + 25\{95\} + 4\{10\ \}$
$+ 5\{10\ 1^2\} + 19\{10\ 2\} + 5\{10\ 21^2\} + 15\{10\ 2^2\} + 28\{10\ 31\} + 32\{10\ 4\}$
$+ 11\{11\ 1\} + \{11\ 1^3\} + 24\{11\ 21\} + 30\{11\ 3\} + 7\{12\ \} + 10\{12\ 1^2\}$
$+ 28\{12\ 2\} + 18\{13\ 1\} + 9\{14\ \}$

$$M_- \otimes \{2^2\} =$$

$\{2^2\} + \{321\} + \{3^21^2\} + \{41^2\} + \{42\} + \{42^2\}$
$+ 2\{431\} + \{4321\} + \{43^2\} + 2\{4^2\} + \{4^22\} + \{4^22^2\}$
$+ \{51\} + \{51^3\} + 3\{521\} + \{53\} + 2\{531^2\} + 3\{532\}$
$+ 2\{53^21\} + 4\{541\} + \{5421\} + 3\{543\} + \{5432\} + 2\{5^2\}$
$+ 3\{5^21^2\} + 2\{5^22\} + 2\{5^231\} + 2\{5^24\} + 2\{61^2\} + 4\{62\}$
$+ 2\{621^2\} + 2\{62^2\} + 7\{631\} + 3\{6321\} + 3\{63^2\} + \{63^22\}$
$+ 5\{64\} + 2\{641^2\} + 8\{642\} + \{642^2\} + 3\{6431\} + 2\{64^2\}$
$+ 8\{651\} + 4\{6521\} + 7\{653\} + 5\{6^2\} + 2\{6^21^2\} + 5\{6^22\}$
$+ 2\{71\} + \{71^3\} + 7\{721\} + \{72^21\} + 6\{73\} + 6\{731^2\}$
$+ 8\{732\} + \{732^2\} + 4\{73^21\} + 13\{741\} + 6\{7421\} + 10\{743\}$
$+ 7\{75\} + 7\{751^2\} + 15\{752\} + 13\{761\} + 5\{7^2\} + \{8\}$
$+ 5\{81^2\} + 7\{82\} + 3\{821^2\} + 6\{82^2\} + 16\{831\} + 7\{8321\}$
$+ 9\{83^2\} + 15\{84\} + 8\{841^2\} + 19\{842\} + 25\{851\} + 15\{86\}$
$+ 5\{91\} + 3\{91^3\} + 14\{921\} + 3\{92^21\} + 12\{93\} + 10\{931^2\}$
$+ 19\{932\} + 30\{941\} + 22\{95\} + \{10\ \} + 8\{10\ 1^2\} + 14\{10\ 2\}$
$+ 7\{10\ 21^2\} + 10\{10\ 2^2\} + 31\{10\ 31\} + 26\{10\ 4\} + 8\{11\ 1\} + 4\{11\ 1^3\}$
$+ 24\{11\ 21\} + 25\{11\ 3\} + 3\{12\ \} + 14\{12\ 1^2\} + 21\{12\ 2\} + 14\{13\ 1\}$
$+ 4\{14\ \}$