“Bosons love to come together; fermions can’t stand each other”
Daniel Kleppner

Bosons, Fermions and Symmetric Functions

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Symmetric Functions

- Newton's *Arithmetica Universalis*
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Monomial Symmetric Functions

- A *symmetric* monomial

$$m_\lambda(x) = \sum_\alpha x^\alpha \tag{1}$$

involves a sum over all distinct permutations α of $(\lambda) = (\lambda_1, \lambda_2, \ldots)$. e.g. if $(x) = (x_1, x_2, x_3)$ then

$$m_{21}(x) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

$$m_{1^3}(x) = x_1 x_2 x_3$$

- If $\lambda \vdash n$ then $m_\lambda(x)$ is homogeneous of degree n. Normally we shall assume (x) involves an infinite number of variables. The monomials form a basis for the *ring of symmetric functions*.
- Other bases exist.

Schur Functions and Monomials

- The Schur-functions ($S-$functions) are indexed by ordered partitions (λ) and are combinatorially defined as

$$s_\lambda(x) = \sum_T x^T \tag{2}$$

where the sum is over all semistandard $\lambda-$tableaux

- In just three variables, $(x) = (x_1, x_2, x_3)$, we have for $(\lambda) = (21)$

$$s_{21}(x_1, x_2, x_3) = x_1^2x_2+x_1^2x_3+x_1x_2^2+x_1x_2x_3+x_1x_2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2$$

corresponding to the eight tableaux

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Schur Functions and Monomials

- In terms of monomials

$$s_{21}(x_1, x_2, x_3) = m_{21}(x_1, x_2, x_3) + 2m_{1^3}(x_1, x_2, x_3)$$

In an arbitrary number of variables $(x) = (x_1, x_2, \ldots)$

$$s_{21}(x) = m_{21}(x) + 2m_{1^3}(x)$$

- Generally,

$$s_\lambda(x) = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu \qquad (3)$$

where $K_{\lambda\mu}$ is the Kostka matrix.

Schur Functions and other Symmetric Functions

- Note that there is a wide choice of the variables (x). If they are chosen as the eigenvalues of unitary matrices of rank N then the $S-$functions become the characters of the covariant representations of the unitary group $U(N)$
- There are two special cases of interest, being closely related to properties of bosons and fermions respectively:-

$s_N = h_N$ The *homogeneous* symmetric functions

$s_{1^N} = e_N$ The *elementary* symmetric functions

Schur Functions and other Symmetric Functions

- The *power sum* symmetric functions, p_r, are simply defined as

$$p_r = \sum x_i^r = m_r(x) \tag{4}$$

and to form a complete basis we need the multiplicative power sums

$$p_\sigma = p_{\sigma_1} p_{\sigma_2} \dots \tag{5}$$

- The characters χ_σ^λ of $S(N)$ provide the link between $S-$functions and power sum symmetric functions.

$$s_\lambda = \sum_\sigma z_\sigma^{-1} \chi_\sigma^\lambda p_\sigma \tag{6}$$

Schur Functions and other Symmetric Functions

- For any partition (σ)

$$z_\sigma = \prod_{i \geq 1} i^{m_i} m_i! \tag{7}$$

where $m_i = m_i(\sigma)$ is the number of parts of σ equal to i.

- We have the two special cases

$$h_n = \sum_{|\sigma|=n} z_\sigma^{-1} p_\sigma \tag{8a}$$

$$e_n = \sum_{|\sigma|=n} \varepsilon_\sigma z_\sigma^{-1} p_\sigma \tag{8b}$$

where

$$\varepsilon_\sigma = \chi_\sigma^{1^n} = (-1)^{|\sigma| - \ell(\sigma)} \tag{9}$$

The $n-$Dimensional Harmonic Oscillator

- The $n-$dimensional harmonic oscillator has the metaplectic group $Mp(2n)$ as its dynamical group and is the double covering group of the non-compact group $Sp(2n,\Re)$.
- Under $Mp(2n) \to Sp(2n,\Re)$

$$\tilde{\Delta} \to \Delta_+ + \Delta_- \tag{10}$$

Harmonic series irreps of $Sp(2n,\Re)$ are labelled $\langle \frac{1}{2}k(\lambda)\rangle$ with $(\lambda) = (\lambda_1, \lambda_2, \ldots)$ for which the conjugate partition $(\tilde{\lambda}) = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$ satisfies the constraints

$$\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq k \qquad \text{and} \qquad \tilde{\lambda}_1 \leq n \tag{11}$$

- $$\Delta_+ \equiv \langle \tfrac{1}{2}(0)\rangle \qquad \text{and} \qquad \Delta_- \equiv \langle \tfrac{1}{2}(1)\rangle \tag{12}$$

The n-Dimensional Harmonic Oscillator

- Under $Sp(2n,\Re) \to U(n)$

$$\Delta_+ \to \varepsilon^{\frac{1}{2}}(\{0\}+\{2\}+\{4\}+\dots) \tag{13a}$$

$$\Delta_- \to \varepsilon^{\frac{1}{2}}(\{1\}+\{3\}+\{5\}+\dots) \tag{13b}$$

- Thus Δ_+ covers the *even* parity states and Δ_- the *odd* parity states

- Succinctly, under $Mp(2n) \to U(n)$

$$\tilde{\Delta} \to \varepsilon^{\frac{1}{2}} M \qquad \text{with} \qquad M = \sum_{m=0}^{\infty}\{m\} \tag{14}$$

The 1-Dimensional Harmonic Oscillator

- The degeneracy group is $U(1)$, but all its irreps are one-dimensional. For N non-interacting particles in a one-dimensional harmonic oscillator we wish to count the number of *symmetric* states for bosons and *antisymmetric* states for fermions. i.e.

$$M \otimes \{N\} = \sum_{k=0}^{\infty} g_N^k \{k\} \qquad \text{bosons} \tag{15a}$$

$$M \otimes \{1^N\} = \sum_{\ell=\frac{N(N-1)}{2}}^{\infty} c_N^\ell \{\ell\} \qquad \text{fermions} \tag{15b}$$

The 1-Dimensional Harmonic Oscillator

- We find g_N^k is the number of partitions of k into at most N parts with repetitions and null parts allowed while c_N^ℓ is the number of partitions of ℓ into N distinct parts, including the null part.
-
$$c_N^\ell = g_N^k \qquad \text{if} \qquad \ell = k + \tfrac{N(N-1)}{2} \tag{16}$$
- Can map one of the sets of partitions onto the other by adding, or subtracting $\rho_N = (N-1, \ldots, 2, 1, 0)$. Adding ρ_N to the partitions of k into at most N parts converts them into partitions, all of whose parts are distinct. and hence the above equivalence.

The 1-Dimensional Harmonic Oscillator

- Thus there is a one-to-one correspondence between the counts of the states formed by N non-interacting bosons and fermions in a one-dimensional harmonic oscillator. Their thermodynamic properties are equivalent apart from a shift in the ground state energy.
- Suppose

$$\tilde{\Delta} \otimes \{N\} = \sum_{k=0}^{\infty} s_N^k \langle \tfrac{N}{2}(k) \rangle \tag{17a}$$

and

$$\tilde{\Delta} \otimes \{1^N\} = \sum_{\ell=\frac{N(N-1)}{2}}^{\infty} a_N^{\ell} \langle \tfrac{N}{2}(\ell) \rangle \tag{17b}$$

The 1-Dimensional Harmonic Oscillator

- then

$$s_N^k = g_N^k - g_N^{k-2} \qquad \text{and} \qquad a_N^\ell = g_N^{\ell - \frac{N(N-1)}{2}} - g_N^{\ell - \frac{N(N-1)}{2} - 2} \quad (18)$$

- Thus the $Sp(2, \Re)$ content of the two plethysms differ simply by

$$\ell \to \ell - \tfrac{N(N-1)}{2} \qquad \text{i.e.} \quad a_N^\ell = s_N^{\ell - \frac{N(N-1)}{2}} \quad (19)$$

which again reflects the boson-fermion symmetry of the one-dimensional harmonic oscillator.

Symmetric Functions and Partition Functions

- Consider an ideal gas of N-noninteracting particles
  Define the canonical partition function of statistical physics as

$$\mathcal{Z}_N(\beta) = \mathcal{T}r\left(e^{-\beta\mathcal{H}}\right) \tag{20}$$

where $\beta = (k_B T)^{-1}$ and

$$\mathcal{H} = \sum_{i=1}^{N} \mathcal{H}_i \tag{21}$$

is the Hamiltonian, the sum of N identical single particle Hamiltonians, with a spectrum of energy eigenvalues $\mathcal{E}_1, \mathcal{E}_2, \ldots$ (with possible degeneracies)

Symmetric Functions and Partition Functions

- For a single particle, boson or fermion,

$$\mathcal{Z}_1(\beta) = \sum_{i=1} e^{(-\beta \mathcal{E}_i)} \tag{22}$$

- Introduce a set of variables, $(x) = (x_1, x_2, \dots)$, not necessarily finite in number, with $x_i = e^{(-\beta \mathcal{E}_i)}$
- Note that $\mathcal{Z}_1(\beta) = s_1(x) = e_1(x) = h_1(x) = p_1(x)$ in such variables.
- For N-noninteracting particles we are interested in symmetrising N copies of the single particle function in the variables (x) which is an $N-$fold plethysm of the appropriate symmetric functions.

Symmetric Functions and Partition Functions

- Recall $p_1(x) \otimes p_r(x) = p_r(x) = \sum x^r = \mathcal{Z}_1(r\beta)$ (for bosons or fermions)
- Furthermore, $s_1(x) \otimes \{\lambda\} = \{\lambda\}(x) = p_1(x) \otimes \{\lambda\}$
- But,

$$s_\lambda = \sum_\sigma z_\sigma^{-1} \chi_\sigma^\lambda p_\sigma \tag{3}$$

- For N fermions we choose $\{\lambda\} = \{1^N\}$ while for bosons $\{\lambda\} = \{N\}$ and are immediately led to

$$\mathcal{Z}_N(\beta)^\pm = \sum_{|\sigma|=N} \varepsilon_\sigma^\pm z_\sigma^{-1} \mathcal{Z}_1(\sigma\beta) \tag{23}$$

Symmetric Functions and Partition Functions

- where $\varepsilon^+ = 1, \ \varepsilon^- = (-1)^{|\sigma|-\ell(\sigma)}$ and

$$\mathcal{Z}_1(\sigma\beta) = \prod_{i=1}^{\ell(\sigma)} \mathcal{Z}_1(\sigma_i\beta) \tag{24}$$

Thus the canonical partition function for N-noninteracting bosons or fermions is completely determined by the single particle partition function. The coefficients sum to unity for bosons (+) and to zero for fermions (−). For example:-

$$\mathcal{Z}_5(\beta)^{\pm} = \frac{1}{120}\big(24\mathcal{Z}_1(5\beta) \pm 30\mathcal{Z}_1(4\beta)\mathcal{Z}_1(\beta) \pm 20\mathcal{Z}_1(3\beta)\mathcal{Z}_1(2\beta)$$

$$+20\mathcal{Z}_1(3\beta)\mathcal{Z}_1(\beta)^2 + 15\mathcal{Z}_1(2\beta)^2\mathcal{Z}_1(\beta) \pm 10\mathcal{Z}_1(2\beta)\mathcal{Z}_1(\beta)^3 + \mathcal{Z}_1(\beta)^5\big)$$

References

"One can measure the importance of a scientific work by the number of earlier publications rendered superfluous by it" David Hilbert

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